1.1 MA 232 Linear Algebra

Please tell me:

NAME

HOMETOWN

MAJOR & INTEREST

CAREER OBJECTIVE

COMPUTER SKILLS

A <u>linear equation</u> in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = b$$

where $a_1, \dots, a_n$ and b are real numbers.

A collection of one or more linear equations involving the same varibles is called a <u>linear system</u> [System of linear equation]

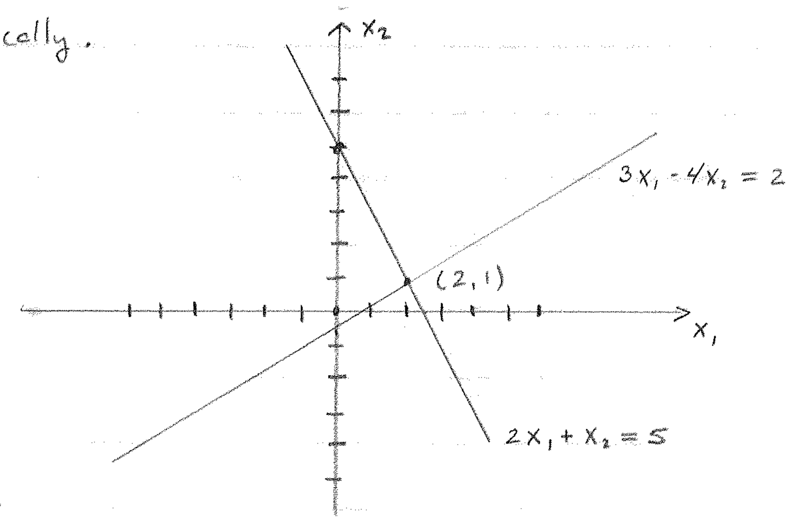
<u>Ex</u>

$$2x_1 + x_2 = 5$$
$$3x_1 - 4x_2 = 2$$

A <u>solution</u> of a linear system is a set of numbers $s_1, \dots, s_n$ which make each equation a true statement when substituted in for the $x_1, \dots, x_n$.

Now, consider what the solution of a linear system is geometrically.

$$2x_1 + x_2 = 5$$
$$3x_1 - 4x_2 = 2$$



Given 2 lines, there can be

1. no points of intersection (parallel)
2. exactly 1 point of intersection
3. infinitely many points of intersection (co-linear)

Same is true for linear systems.

1. no solution
2. exactly one solution
3. infinitely many solutions.

Matrix Notation

A <u>matrix</u> is a rectangular array of numbers

We can use matrices to simplify the task of computing the solution to a linear system.

Consider the system: $\begin{cases} x_1 + 2x_2 = 7 \\ 3x_1 + 8x_2 = 25 \end{cases}$

One way to solve this is to solve the first eq. for x_1 and substitute this value into the second eq. $x_1 = 7 - 2x_2$

so

$$3(7-2x_2) + 8x_2 = 25$$
$$21 - 6x_2 + 8x_2 = 25$$
$$2x_2 = 4$$
$$x_2 = 2$$

now determine x_1: $x_1 = 7 - 2(2) = 7 - 4 = 3$

So the solution is $\boxed{x_1 = 3,\ x_2 = 2}$.

Another way is to use the x_1 in the first eq. to eliminate the x_1 in the second eq.

$x_1 + 2x_2 = 7$ given

$-3(x_1 + 2x_2) = -3(7)$ mult. principle

$-3x_1 - 6x_2 = -21$ distrib.

also $\quad 3x_1 + 8x_2 = 25 \quad$ given

$$\begin{array}{r} + \; (-3x_1 - 6x_2 = -21) \\ \hline 0x_1 + 2x_2 = 4 \end{array}$$

Now replace the second eq. in our system by this eq.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ 2x_2 &= 4 \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 + 2x_2 &= 7 \\ x_2 &= 2 \end{aligned}$$

Notice, however that the new second eq can be used to eliminate the x_2 in the first eq...

$$\begin{array}{r} x_1 + 2x_2 = 7 \\ -\; 2x_2 = -4 \\ \hline x_1 + 0x_2 = 3 \end{array}$$

So we have

$$\begin{aligned} x_1 \quad &= 3 \\ x_2 &= 2 \end{aligned}$$

and we are done.

Lets try another:
$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \quad (1) \\ -x_1 + 3x_2 - x_3 &= 2 \quad (2) \\ 2x_1 \qquad - 3x_3 &= -17 \quad (3) \end{aligned}$$

(1) + (2) $\rightarrow$ (2)
$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \\ x_2 + 3x_3 &= 17 \\ 2x_1 \qquad -3x_3 &= -17 \end{aligned}$$

$-2 \text{①} + \text{③} \rightarrow \text{③}$

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \\ x_2 + 3x_3 &= 17 \\ 4x_2 - 11x_3 &= -47 \end{aligned}$$

$-4 \text{②} + \text{③} \rightarrow \text{③}$

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \\ x_2 + 3x_3 &= 17 \\ -23x_3 &= -115 \end{aligned}$$

$-\frac{1}{23} \text{③} \rightarrow \text{③}$

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \\ x_2 + 3x_3 &= 17 \\ x_3 &= 5 \end{aligned}$$

$-3 \text{③} + \text{②} \rightarrow \text{②}$

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 15 \\ x_2 &= 2 \\ x_3 &= 5 \end{aligned}$$

$-4 \text{③} + \text{①} \rightarrow \text{①}$

$$\begin{aligned} x_1 - 2x_2 &= -5 \\ x_2 &= 2 \\ x_3 &= 5 \end{aligned}$$

$2 \text{②} + \text{①} \rightarrow \text{①}$

$$\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= 5 \end{aligned}$$

Solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 5$

Finally we use matrix form

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 15 \\ -1 & 3 & -1 & 2 \\ 2 & 0 & -3 & -17 \end{array}\right]$$

This is called an <u>augmented matrix</u> because it has been augmented by the RHS values

$$\begin{array}{c} \\ R1 + R2 \\ -2R1 + R3 \end{array} \begin{bmatrix} 1 & -2 & 4 & 15 \\ 0 & 1 & 3 & 17 \\ 0 & 4 & -11 & -47 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ -4R2 + R3 \end{array} \begin{bmatrix} 1 & -2 & 4 & 15 \\ 0 & 1 & 3 & 17 \\ 0 & 0 & -23 & -115 \end{bmatrix}$$

$$\begin{array}{c} -4R3 + R1 \\ -3R3 + R2 \\ \frac{R3}{-23} \end{array} \begin{bmatrix} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

solution

$$\begin{array}{c} 2R2 + R1 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$x_1 = -1$

$x_2 = 2$

$x_3 = 5$

We've been doing <u>elementary row operations</u>

1. Any row can be replaced by the sum of that row and a constant multiple (REPLACEMENT)

2. Any two rows may be interchanged (INTERCHANGE)

3. Any row may be replaced by a nonzero multiple of itself (SCALING)

Two matrices are <u>row equivalent</u> if one can be obtained from the other via elementary row operations.

Row equivalent augmented matrices represent linear systems with the same solution.

Finally we consider the question of existence.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -2x_1 + 4x_2 &= -1 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 - 2x_2 &= 1 \\ 0 + 0 &= +1 \end{aligned} \leftarrow \text{? does zero} = 1\text{?}$$

look at 2nd eq. – LHS is const mult. of LHS of eq 1 but RHS are not related the same way.

This system is <u>inconsistant</u>.

– linear systems which are inconsistant have no solution.