1.2 Row Reduction and Echelon Forms

A rectangular matrix is in <u>echelon form</u> ("steplike") if it has the following three properties

1. All nonzero rows are above rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero. (consequense of #2).

It is in <u>reduced echelon form</u> if the two additional conditions are met:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

<u>Ex</u> $\begin{bmatrix} \bullet & x & x & x \\ 0 & \bullet & x & x \\ 0 & 0 & \bullet & x \\ 0 & 0 & 0 & \bullet \end{bmatrix}$

<u>Ex</u> $\begin{bmatrix} \bullet & x & x & x & x & x \\ 0 & 0 & \bullet & x & x & x \\ 0 & 0 & 0 & \bullet & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\bullet$ is a <u>pivot pos.</u>

Ex $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ex $\begin{bmatrix} 1 & x & 0 & 0 & x & x \\ 0 & 0 & 1 & 0 & x & x \\ 0 & 0 & 0 & 1 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

NOTE: While a given matrix usually has more than one corresponding echelon form, it has only one reduced echelon form

Theorem 1: Each matrix is row equivalent to one and only one reduced echelon matrix.

Row Reduction Algorithm

1. The pivot column is the left-most nonzero column.
2. Interchange rows if necessary to move a nonzero entry into the top position, the <u>pivot position</u>
3. Use elementary row operations to create zeros in all positions below the pivot.
4. Cover the row and column which contain the pivot and apply steps 1 – 3 to the resulting unreduced submatrix

This yields a matrix in <u>echelon form</u> which is row equivalent to the original matrix.

5. Beginning with the rightmost pivot, scale if necessary so that the pivot is 1 and use the pivot to create zeros in all positions above the pivot. Repeat with all other pivots, moving to the left.

This yields a matrix in <u>reduced echelon form.</u>

Ex

Pivot position

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

need to switch with another row so pivot is nonzero

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \begin{array}{l} R4 \\ \\ \\ R1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \begin{array}{l} \\ R1+R2 \\ 2R1+R3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \frac{5}{2}R2+R3 \\ -\frac{3}{2}R2+R4 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R4 \\ R3 \end{array}$$

Echelon Form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R3/(-5) \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} (-9)R3+R1 \\ (-6)R3+R2 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ R2/2 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} (-4)R2+R1 \\ \\ \\ \\ \end{array}$$

Reduced Echelon Form

Solutions of Linear Systems

Consider the system represented by the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

which is

$$\begin{aligned} x_1 \qquad\qquad &= 5 \\ x_2 \qquad &= -1 \\ x_3 &= 2 \end{aligned}$$

Obviously, there is only one unique solution here.

Now consider
$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \quad + 2x_3 &= -1 \\ x_2 + 3x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

Here the last eq., while true, conveys incomplete information regarding x_1, x_2 & x_3. In fact, this system has an infinite number of solutions.

We call the varibles which appear in only one eq. <u>basic variables</u>, or <u>leading variables</u>. All other varibles are called <u>free variables</u>.

$$\begin{cases} x_1 = -1 - 2x_3 \\ x_2 = 5 - 3x_3 \\ x_3 = \text{anything (free)} \end{cases}$$

general solution

We are free to choose any value we desire for x_3, but then x_1 and x_2 are fixed. Every different choice of x_3 results in a different solution.

Ex Find the general soln. of the system corresponding to

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} : \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 0x_2 = -2$$
$$0x_1 + x_2 = 3$$

$$\boxed{x_1 = -2, \quad x_2 = 3}$$

Ex Find the general solution of the system

$$x_1 + 2x_2 \qquad + 3x_4 = 5$$
$$x_3 - x_4 = 2$$

basic variables : x_1, x_3
free variables : x_2, x_4

$$\begin{cases} x_1 = 5 - 2x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 2 + x_4 \\ x_4 \text{ is free} \end{cases}$$

The soln. is <u>parametrized</u> in terms of the free variables. <u>General Soln in parametric form.</u>

Some thoughts on numerical computations...

Even non-computer folks should be aware of some of the ways computers do L.A.

- "Elegant textbook method" often not numerically efficient.
- Computers are finite number crunchers and are subject to introduced error in every calculation.

1. Partial pivoting - before performing a pivot operation, switch the pivot row with the row below it with the numerically largest element element in the pivot column. Helps to avoid dividing a large number by a small number.

2. In practice, transforming an augmented matrix into reduced row echelon form is not used to solve large systems. In fact, if elimination is used at all it is only to get echelon form. Then a "backward" solve is used.

HOMEWORK COMMENT

Consistent - what does it mean?

1.2 #22. $\begin{bmatrix} 1 & 4 & 2 \\ -3 & h & -1 \end{bmatrix}$ What does h need to be for this to be consistent?

$\begin{bmatrix} 1 & 4 & 2 \\ 0 & h+12 & 5 \end{bmatrix}$

this entry cannot be zero, else the system would be inconsistent → $\boxed{h \neq -12}$

#23. $\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & h+8 \end{bmatrix}$ $\boxed{\text{Consistent for any } h}$

#24. $\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6+3h & -2-h \end{bmatrix}$ cannot have $6+3h=0$ and $-2-h \neq 0$. If $h=-2$ then $6+3h=0$ but so does $-2-h$. $\boxed{\text{Any } h}$

#25. $\begin{bmatrix} 1 & h & 3 \\ 2 & 8 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & h & 3 \\ 0 & 8-2h & -5 \end{bmatrix}$ need $8-2h \neq 0$ $\boxed{h \neq 4}$

1.1 #28 $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$

$\boxed{\text{h can be any number}}$

$\begin{bmatrix} 1 & 4 & -2 \\ 0 & h-12 & 0 \end{bmatrix}$