1.3 Vectors in $\mathbb{R}^n$

A matrix with only one column is called a <u>column vector</u>. We usually shorten this to just <u>vector</u>. Some examples are

$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

and we use the notation $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$ to indicate a vector (text uses bold letters). Some people prefer $\underline{u}$ to get away from the "directed line segment" idea.

The order of <u>entries</u> in a vector is important, thus $\vec{u} = \vec{v}$ implies that $\vec{u}$ and $\vec{v}$ have the same entries in the same order. Of course, this also means they are the same length.

If a vector has 2 real entries it is in $\underline{\mathbb{R}^2}$, the set of all vectors with 2 real entries

Addition: $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$ $\vec{v} = \begin{bmatrix}5\\-1\end{bmatrix}$

$$\vec{u} + \vec{v} = \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}5\\-1\end{bmatrix} = \begin{bmatrix}1+5\\2-1\end{bmatrix} = \begin{bmatrix}6\\1\end{bmatrix}$$

or if we take the more general case

$$\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix} \quad \vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}$$ which is also in $\mathbb{R}^2$.

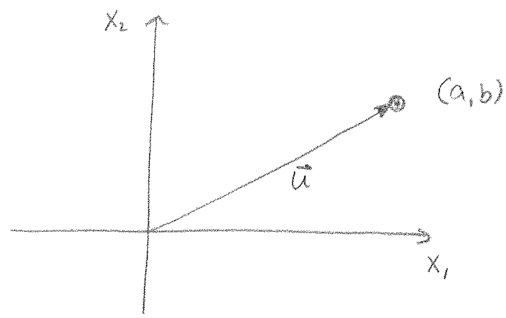
We also can obtain the scalar multiple of a vector. $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$c\vec{u} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \cdot 1 \\ c \cdot 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} \qquad \text{also in } \mathbb{R}^2$$

<u>Ex</u> $\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$\vec{u} + 3\vec{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$$

Geometric Interpretation



$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$

We can also think of $\vec{u}$ as a directed line segment from the origin $(0,0)$ to the point (a,b).

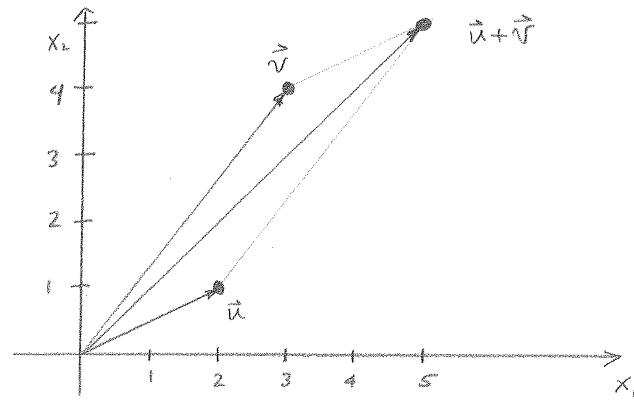
In fact, we often will think of $\vec{u}$ as a point, not a directed line segment. Thus (a,b) and $\begin{bmatrix} a \\ b \end{bmatrix}$ can be understood to be equivalent.

The point $(0,0)$ can also be described $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to which we give the notation

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{"the zero vector".}$$

Vectors in $\mathbb{R}^2$ can be added geometrically

$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



<u>Parallelogram rule for addition:</u>
if $\vec{0}$, $\vec{u}$ and $\vec{v}$ are 3 points in a plane ($\mathbb{R}^2$) then $\vec{u}+\vec{v}$ is also in the plane and is the 4^{th} vertex of the parallelogram whose other 3 vertices are $\vec{0}, \vec{u}, \vec{v}$.

Consider $c\vec{u}$ where c is any scalar. This product describes the set of points (vectors in $\mathbb{R}^2$) on the line passing through $\vec{0}$ and $\vec{u}$.

Vectors in $\mathbb{R}^3$ are essentially points in 3-dimensional space.

<u>Ex</u> $\quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad$ (3 entries)

They can be added geometrically with a "parallelepiped" law.

In general, we can talk about vectors in $\mathbb{R}^n$, these have n entries.

Algebraic Properties of $\mathbb{R}^n$

For all $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ and all scalars c and d:

i)	$\vec{u}+\vec{v}=\vec{v}+\vec{u}$	communitive
ii)	$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$	associative
iii)	$\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$	additive identity
iv)	$\vec{u}+(-\vec{u})=-\vec{u}+\vec{u}=\vec{0}$	additive inverse
v)	$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$	distributive
vi)	$(c+d)\vec{u}=c\vec{u}+d\vec{u}$	distributive
vii)	$c(d\vec{u})=(cd)(\vec{u})$	associative
viii)	$1\vec{u}=\vec{u}$	multiplicative identity

Given a collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p$, we can construct a <u>linear combination</u> of the $\vec{v}_i$ using the c_i

$$\vec{y}=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_p\vec{v}_p$$

THIS IS IMPORTANT - TAPE IT ABOVE
YOUR SINK AND THINK ABOUT IT WHILE
YOU BRUSH YOUR TEETH!

If $\vec{v}_i$, $i=1\dots p$, are vectors in $\mathbb{R}^n$, then the set of all possible linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ is called <u>the subset of $\mathbb{R}^n$ spanned by $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$</u> and is denoted by

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$$

Ex Show that $\vec{b}=\begin{bmatrix}10\\24\end{bmatrix}$ is in the plane spanned by $\vec{a}_1=\begin{bmatrix}3\\2\end{bmatrix}$ and $\vec{a}_2=\begin{bmatrix}1\\5\end{bmatrix}$

For $\vec{b}$ to be in span$\{\vec{a}_1, \vec{a}_2\}$, there must be 2 constants, x_1 and x_2, for which

$$x_1\begin{bmatrix}3\\2\end{bmatrix}+x_2\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}10\\24\end{bmatrix}$$

$$\begin{bmatrix}3x_1\\2x_1\end{bmatrix}+\begin{bmatrix}x_2\\5x_2\end{bmatrix}=\begin{bmatrix}10\\24\end{bmatrix}$$

$$\begin{bmatrix}3x_1+x_2\\2x_1+5x_2\end{bmatrix}=\begin{bmatrix}10\\24\end{bmatrix}$$

therefore
$$\begin{aligned}3x_1+x_2&=10\\2x_1+5x_2&=24\end{aligned}$$
Ahhh...

we see that if values for x_1 and x_2 can be found (i.e. a consistant linear system) then $\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2\}$.

$$\begin{bmatrix}3&1&10\\2&5&24\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&4\end{bmatrix}$$

so $x_1=2$, $x_2=4$.

$\therefore$ $\vec{b}$ is in span$\{\vec{a}_1, \vec{a}_2\}$

1.3

The vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + \vec{x}_p\vec{a}_p = \vec{b}$ has the same solution set as the linear system whose augmented matrix is $[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_p \ \vec{b}\,]$.

This is a row vector (or row matrix) whose entries are column vectors. If each $\vec{a}_i$ has n elements (in $\mathbb{R}^n$) then this matrix will be $n \times (p+1)$.