1.4 The Equation $A\vec{x} = \vec{b}$

Recall that if we consider the columns of an $m \times n$ matrix A to be vectors denoted $\vec{a}_i$, $i = 1, \dots, n$; then we can say

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

We saw before that we can form a linear combination of the columns of A

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + \dots + x_n \vec{a}_n = \vec{b}$$

where $\vec{b} \in \mathbb{R}^m$ as are the $\vec{a}_i$.

We <u>define</u> matrix-vector multiplication so that it computes exactly this linear combination

$$A\vec{x} = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n$$

Notice that the length of $\vec{x}$ must be the same as the number of columns in A.

We usually use uppercase letters for matricies

Ex $A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0-2 \\ 6+3+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$$

Ex Consider the linear system

$$2x_1 \quad - x_3 = 0$$
$$6x_1 + x_2 + 3x_3 = 15$$

$$\begin{bmatrix} 2x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$$

$$x_1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$$

$A\vec{x} = \vec{b}$ if $A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 1 & 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

We call this a <u>matrix eq.</u>
Just as in the augmented matrix related to a linear system, we see here that A is the matrix of coefficients and $\vec{b}$ is the vector of RHS values. The vector $\vec{x}$ is the vector of unknowns.

Theorem 1

Given an $m \times n$ matrix A with columns $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$, and given $\vec{b}$ in $\mathbb{R}^m$, the matrix equation $A\vec{x} = \vec{b}$ has the same solution set as the vector equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\vec{a}_1 \ \ \vec{a}_2 \ \cdots \ \vec{a}_n \ \ \vec{b}].$$

Corollary

The equation $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is a linear combination of the columns of A.

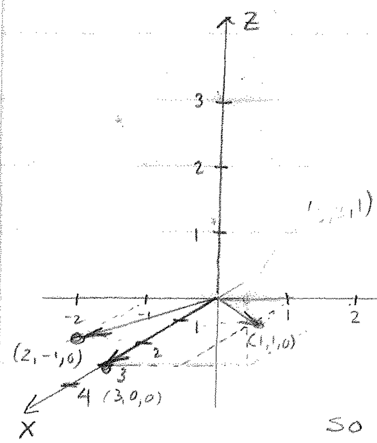
We now return to the concept of <u>span</u>. Recall that $\text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is the set of all vectors which are linear combinations of the $\vec{a}_i$ vectors.

If $\vec{a}_i$ are the columns of a matrix A then we can use the notation $\text{Span}\{A\}$ to mean $\text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$.

So, we can reword the corollary to say:

"The equation $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is in $\text{span}\{A\}$."

Ex Consider a 3-dimensional coordinate system.



The point (3,1,2) is located using a linear combination of the unit vectors

$$3\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix}$$

so $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix}$ has the solution $\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Suppose we tried to find $\vec{x}$ such that

$$\begin{bmatrix}1&2&3\\1&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\1\\2\end{bmatrix} \quad ?$$

First, what is the augmented matrix? $\begin{bmatrix}1&2&3&3\\1&-1&0&1\\ \boxed{0\ \ 0\ \ 0\ \ 2}\end{bmatrix}$

inconsistant

But what is also going on is this

$$x_1\begin{bmatrix}1\\1\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-1\\0\end{bmatrix} + x_3\begin{bmatrix}3\\0\\0\end{bmatrix} = \ ?$$

here the column vectors can also be thought of as points in the xy-plane since none of them have any extent in the z direction.

It is easy to see that there is no possible way to get $\begin{bmatrix}3\\1\\2\end{bmatrix}$ from these 3 vectors

Theorem 2

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. (all true or all false):

a) For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ is consistant,

b) The columns of A span $\mathbb{R}^m$ ($\mathbb{R}^m = \text{span}\{A\}$),

c) A has a pivot position in every row.

d) Each $\vec{b} \in \mathbb{R}^n$ is a linear combination of columns in A

We can compute $A\vec{x}$ two ways:

1) as a linear combination of the columns $\vec{a_i}$ of A using the weights x_i

2) "row by vector" - The $i^{\underline{th}}$ entry in the product is the sum of the element-by-element products of the $i^{\underline{th}}$ row of A and the vector $\vec{x}$.

Ex

$$\begin{bmatrix} 3 & 6 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6+18+5 \\ 0+3+10 \\ 2+9+10 \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ 21 \end{bmatrix}$$

or

$$\begin{bmatrix} \boxed{3 \quad 6 \quad 1} \\ 0 \quad 1 \quad 2 \\ 1 \quad 3 \quad 2 \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix} 2 \\ 3 \\ 5 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 6 \cdot 3 + 1 \cdot 5 \\ 0 \cdot 2 + 1 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 2 + 3 \cdot 3 + 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ 21 \end{bmatrix}$$

Theorem 3: If A is $m \times n$, $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^n$ and c is a scalar then

a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

b) $A(c\vec{u}) = c(A\vec{u})$