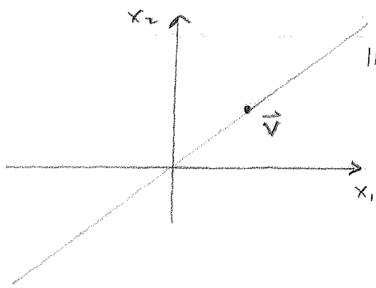
1.5 Solution Sets of Linear Systems

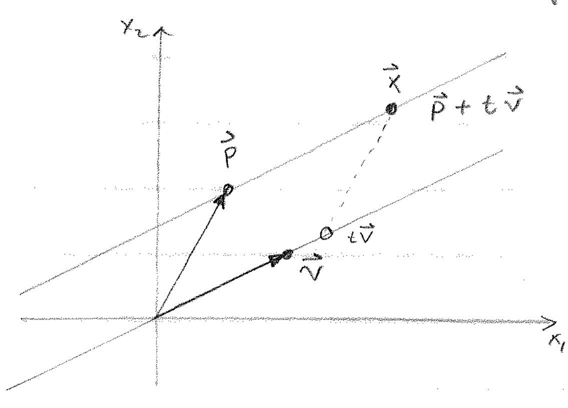
Recall that $t\vec{v}$ is a family of vectors each lying on a line passing through $\vec{0}$ and $\vec{v}$.



line is set of points $t\vec{v}$ for all t.

What, then, is $\vec{p} + t\vec{v}$?

$$\vec{x} = \underbrace{\vec{p}}_{\text{offset or displacement vector}} + \overbrace{t\vec{v}}^{\text{line through } \vec{0} \text{ and } \vec{v}}$$



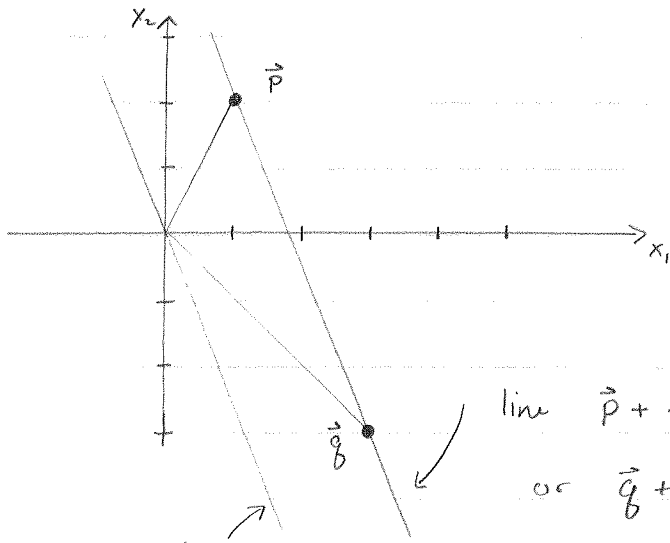
So we see that $\vec{p} + t\vec{v}$ describes the set of points which make up the line $\parallel$ to $t\vec{v}$ and which passes through $\vec{p}$.

We say $\vec{x} = \vec{p} + t\vec{v}$ is the parametric eq. of the line.

<u>Ex</u> Find the parametric eq of the line parallel to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and passing through $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

<u>Ex</u> If $\vec{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, what is a parametric eq. of the line passing through these points?



line $\vec{p} + t(\vec{p} - \vec{q})$

or $\vec{q} + t(\vec{p} - \vec{q})$

<u>NOT UNIQUE</u>

line $t(\vec{p} - \vec{q})$

or $t(\vec{q} - \vec{p})$

Homogeneous Linear Systems

A linear system is <u>homogeneous</u> if it can be written in the form $A\vec{x} = \vec{0}$ where A is $m \times n$ and $\vec{0}$ is the zero vector in $\mathbb{R}^m$.

The system $A\vec{x} = \vec{0}$ always has the solution $\vec{x} = \vec{0}$, which is called the <u>trivial solution.</u> Sometimes this system has a <u>nontrivial solution</u>, i.e. $\vec{x} \neq \vec{0}$.

→ What can we say about the system $A\vec{x} = \vec{0}$?
i.e., when will it have a nontrivial solution?

Thm 1 from section 1.2 says that a consistent linear system has either

a) a unique solution and no free variables

or b) infinitely many solutions and at least one free variable.

We conclude, then, that $A\vec{x}=\vec{0}$ has a non-trivial solution iff the system has at least one free variable. In this case it has infinitely many solutions.

Prove that if $A\vec{x}=\vec{0}$ has a nontrivial solution $\vec{p}$ that it has infinitely many nontrivial solutions.

$$A(c\vec{p}) = c\,A\vec{p} = c\vec{0} = \vec{0}$$

so if $\vec{p}$ is a soln then any scalar multiple of $\vec{p}$ is also a solution.

<u>Ex</u>

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_2 - 5x_3 &= 0 \\ 2x_1 + 4x_2 - 9x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -5 \\ 2 & 4 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -5 \\ 0 & 6 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

(note - don't need column of zeros)

So
$$\begin{cases} x_1 = -\frac{1}{2}x_3 \\ x_2 = \frac{5}{2}x_3 \\ x_3 = x_3 \quad \text{(free)} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ \frac{5}{2}x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix}$$

So the solution $\vec{x}$ is the span $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right\}$

a 1-dimensional space - a line in $\mathbb{R}^3$.

Ex Describe the soln. set in $\mathbb{R}^3$ of $x_1 - 3x_2 + 5x_3 = 0$.

$x_1 = 3x_2 - 5x_3$

$x_2 = x_2$ (free)

$x_3 = x_3$ (free)

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

So the solution of $x_1 - 3x_2 + 5x_3 = 0$ is the set of points on the plane spanned by $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$.

Check: cross product should yield a scalar mult of $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$

Normal vector

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ -5 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j}(3) + \hat{k}(+5) = \hat{i} - 3\hat{j} + 5\hat{k} \text{ or } \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}.$$

How about $x_1 - 3x_2 + 5x_3 = 2$?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + 3x_1 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}_{\text{same plane as before, now}}$$

"shifted" by $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

like $\vec{p} + t\vec{v}$ only now $\vec{p} + t\vec{u} + s\vec{v}$

Theorem 4:

Suppose that $A\vec{x} = \vec{b}$ is consistant for some $\vec{b}$ and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors $\vec{w} = \vec{p} + \vec{v}_h$ where $\vec{v}_h$ is any solution of $A\vec{x} = \vec{0}$.

If $\vec{b} \neq \vec{0}$ then $A\vec{x} = \vec{b}$ is <u>nonhomogeneous</u>. If a nonhomogeneous system has more than one solution, then all the solutions can be written in parametric form.

We saw this for $x_1 - 3x_2 + 5x_2 = 2$. This is a nonhomogeneous L.S.

$$\begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

the solutions of which are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}}_{\text{Solution of homogeneous system}}$$

"particular" solution of nonhomogeneous system.

From Thm 4: if $\vec{w} = \vec{p} + \vec{v}_h$ where $A\vec{p} = \vec{b}$ and $A\vec{v}_h = \vec{0}$

$$A\vec{w} = A(\vec{p} + \vec{v}_h) = A\vec{p} + A\vec{v}_h = \vec{b} + \vec{0} = \vec{b}$$

So $\vec{w}$ is also a solution of $A\vec{x} = \vec{b}$.