1.6 Applications of Linear Systems

Three applications: 1) Economic model, 2) Balancing Chemical Eq. 3) Network flow

We'll look at economic models a little later...

Balancing Chemical Eq.

TNT and oxygen combine (explosively!) in the following manner

$$C_7H_5N_3O_6 + O_2 \longrightarrow CO_2 + H_2O + N_2$$

This is an unbalanced eq. We want to determine the correct amounts of each molecule so that the number of atoms of each element is conserved.

Let our vectors represent Carbon, Hydrogen, Nitrogen and Oxygen so $C_7H_5N_3O_6$ is

$$\begin{bmatrix} 7 \\ 5 \\ 3 \\ 6 \end{bmatrix}$$

and CO_2 is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$

Then we seek x_1, x_2, x_3, x_4 & x_5 such that

$$x_1 \begin{bmatrix} 7 \\ 5 \\ 3 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 7 & 0 & -1 & 0 & 0 \\ 5 & 0 & 0 & -2 & 0 \\ 3 & 0 & 0 & 0 & -2 \\ 6 & 2 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 0 & -7/2 \\ 0 & 0 & 1 & 0 & -14/3 \\ 0 & 0 & 0 & 1 & -5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = \frac{2}{3} x_5$

$x_2 = \frac{7}{2} x_5$

$x_3 = \frac{14}{3} x_5$

$x_4 = \frac{5}{3} x_5$

$x_5 =$ free

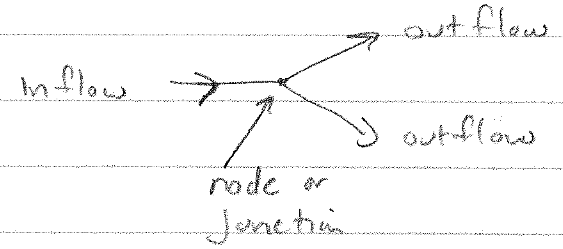
It's convenient to use integers for x_1 through x_5 so lets take $x_5 = 6$. This gives

$x_1 = 4, \ x_2 = 21, \ x_3 = 28,$
$x_4 = 10, \ x_5 = 6$

$$4\, C_7H_5N_3O_6 + 21\, O_2 \to 28\, CO_2 + 10\, H_2O + 6\, N_2$$

Network Flow

Ex Current in wires, traffic, fluid in pipes (sprinklers)

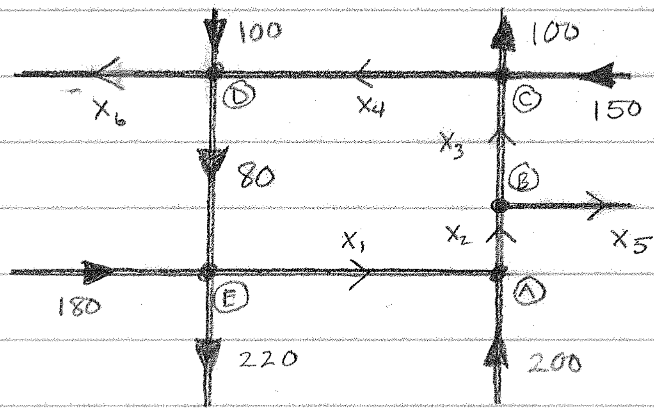


Two Conservation laws

1. inflow to a junction must equal outflow from that same junction

2. Total flow into network equals total flow out of network

Applications of Linear Systems



(A) $X_1 + 200 = X_2$

(B) $X_2 = X_3 + X_5$

(C) $X_3 + 150 = X_4 + 100$

(D) $X_4 + 100 = X_6 + 80$

(E) $180 + 80 = X_1 + 220$

Also

$$100 + 180 + 200 + 150 = X_5 + 100 + X_6 + 220$$

$$\begin{aligned}
X_1 - X_2 \quad\quad\quad\quad\quad\quad &= -200\\
X_2 - X_3 \quad - X_5 \quad &= 0\\
X_3 - X_4 \quad\quad &= -50\\
X_4 \quad - X_6 &= -20\\
X_1 \quad\quad\quad\quad &= 40\\
X_5 + X_6 &= 310
\end{aligned}$$

or

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} -200 \\ 0 \\ -50 \\ -20 \\ 40 \\ 310 \end{bmatrix}$$

This reduces to
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 40 \\ 240 \\ -70 \\ -20 \\ 310 \\ 0 \end{bmatrix}$$

$x_1 = 40$ $\qquad$ $x_4 = x_6 - 20$

$x_2 = 240$ $\qquad$ $x_5 = 310 - x_6$

$x_3 = x_6 - 70$ $\qquad$ $x_6 = \text{free}$

or
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 40 \\ 240 \\ -70 \\ -20 \\ 310 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

1. Note that x_1 and x_2 are independent of x_6
   How does this fit with our diagram?

2. Assuming all $x_i \geq 0$ we need $x_6 \geq 70$ and $x_6 \leq 310$
   so $70 \leq x_6 \leq 310$

NOTE: Correction to 1.6 problem 11. (pg 64)

Bottom of diagram should show
80 flowing <u>in</u> not out.