1.7 Linear Independence

<u>DEF.</u> An indexed set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be <u>linearly independent</u> if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0} \qquad (1)$$

has only the trivial solution. The set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is said to be <u>linearly dependent</u> if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0} \qquad (2)$$

<u>linear dependence relation</u>

<u>Ex</u> Are $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

linearly independent?

If so, then (1) will have only the trivial solution. Examine the corresponding augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3$
$x_2 = x_3$

So that x_3 is free. However, this means that x_3 does not need to be zero and (1) is not only solved by the trivial solution. Thus the 3 vectors are not linearly independent. In fact, if $x_3 = 1$

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{from (2)}$$

Since we can think of $A\vec{x}$ as a linear combination of the columns of A, if $A\vec{x}=\vec{0}$ has a nontrivial solution then the columns of A must be linearly dependent.

So The columns of a matrix A are linearly independent iff the equation $A\vec{x}=\vec{0}$ has only the trivial solution.

If we solve $A\vec{x}=\vec{0}$ by reduction to echelon form and find that there are no free variables [there is a pivot in every column of A] then we know that there is only one solution, the trivial solution, and so the columns of A are linearly independent.

So The columns of a matrix A are linearly independent if every column of A contains a pivot.

For a moment, let's just consider a set of two vectors: $\{\vec{v}_1, \vec{v}_2\}$

Q: when are two vectors linearly dependent?

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \implies c_1\vec{v}_1 = -c_2\vec{v}_2$$
$$\implies \vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$$

A: $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent when they are scalar multiples of each other.

<u>Thm</u> An indexed set $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.

Ex. $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We see that $\vec{v}_2 = 2\vec{v}_1 + 0\vec{v}_3$ so $-2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 = \vec{0}$ which means the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

Note, however, that $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$. The words "at least one" in the statement of the theorem are important.

<u>Thm</u> Any set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p>n$.

In other words, if there are more vectors than entries in the vectors then the set of vectors must be linearly dependent.

<u>Proof</u> Consider $A\vec{x}=\vec{0}$ with A being the $n \times p$ matrix $A=[\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_p]$. If $p>n$ then there are more columns than rows. We cannot have more pivots than rows so $\#\text{pivots} \le n$. This means that at least one column does not have a pivot so A has linearly dependent columns.

Thm If a set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector then S is linearly dependent.

Proof Let $\vec{v}_i = \vec{0}$ be the zero vector. Then

$$0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_{i-1} + \underset{\substack{\uparrow \\ \text{nonzero}}}{1}\vec{v}_i + 0\vec{v}_{i+1} + \cdots + 0\vec{v}_p = \vec{0}$$

so S is linearly dependent.