1.8 Introduction to Linear Transformations

Yet another way to view matrix multiplication ...

$A\vec{x} = \vec{b}$ — here A is an "operator" which describes a mapping from the "space" $\vec{x}$ lives in to the space where $\vec{b}$ resides.

If A is $m \times n$ then $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$ so A describes a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or a subspace of $\mathbb{R}^m$).

<u>Ex</u>

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{a vector from } \mathbb{R}^3} = \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{\text{a vector in } \mathbb{R}^2}$$

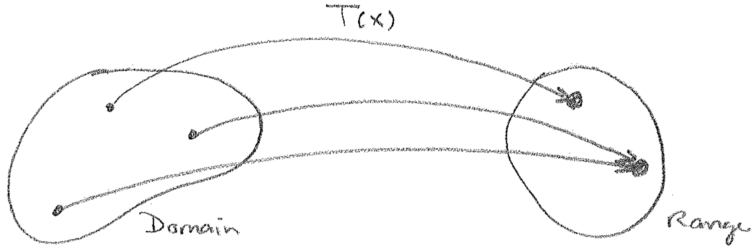
We are describing a <u>function</u> – a rule that transform an element in the <u>Domain</u> space to an element in the <u>range</u> space

Functions are also called <u>Transformations</u> or <u>Mappings</u>

Ex $T: \mathbb{R}^n \to \mathbb{R}^m$

The transformation $T(\vec{x})$ maps a vector to its <u>image</u>.

The question of solving $T(\vec{x}) = \vec{b}$ (really $A\vec{x} = \vec{b}$) is to ask "What element of the domain gets mapped to $\vec{b}$ by T?"



Notice that just as for real variables, the transformation does not have to be one-to-one – however, each element in the domain must have only one image in the range

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ here any vector in $\mathbb{R}^3$ $\begin{bmatrix} a \\ b \\ x \end{bmatrix}$ is mapped to $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for any x.

Ex $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 3 & -8 \end{bmatrix}$ $\bar{b} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$ If $T(\vec{x}) = A\vec{x}$ find $\vec{x}$ s.t. its image under T is $\vec{b}$. Is $\vec{x}$ unique?

Solve $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 3 & 1 & 9 \\ 0 & 3 & -8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & 3 & -1 \\ 0 & 3 & -8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \text{so} \quad \vec{x} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

yes, this uniqu
only solution of
$A\vec{x} = \vec{b}$.

Linear Transformations

If a transformation has the properties

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall\ \vec{u},\vec{v}$ in domain of T

2) $T(c\vec{u}) = cT(\vec{u}) \quad \forall\ \vec{u}$, and scalars c

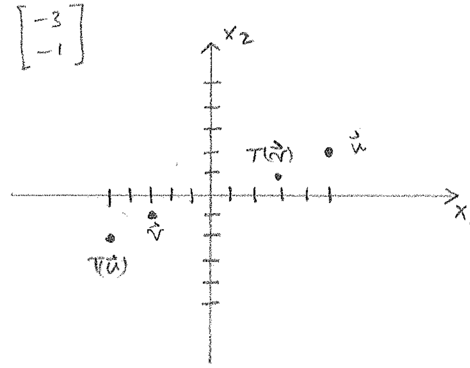
Then it is a linear transformation (T is a linear operator).

→ Recall from Thm 3 that the matrix-vector mult. $A\vec{x}$ is a linear transformation.

Ex #13. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Let $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^2$

a) plot $\vec{u}, \vec{v}, T(\vec{u}), T(\vec{v})$

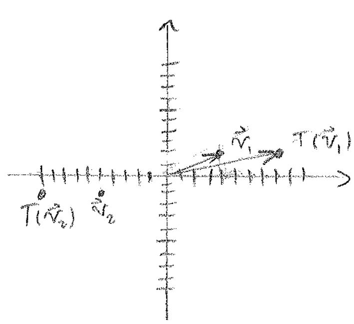


$T(\vec{u}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$

$T(\vec{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) Geometric description of T: reflect through origin

Ex #18 $T(x) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$



$T(\vec{v}_1) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \quad T(\vec{v}_2) = \begin{bmatrix} -10 \\ -2 \end{bmatrix}$

scales in x direction

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Ex #21 $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (unit vectors)

$\vec{y}_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ $\vec{y}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ maps $\vec{e}_1$ onto $\vec{y}_1$ and $\vec{e}_2$ onto $\vec{y}_2$

Find image of $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$T(\vec{e}_1) = \vec{y}_1$: $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$1\vec{a}_1 + 0\vec{a}_2 = \vec{a}_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$T(\vec{e}_2) = \vec{y}_2$: $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

$0\vec{a}_1 + 1\vec{a}_2 = \vec{a}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

So

$A = \begin{bmatrix} 3 & -2 \\ -5 & 7 \end{bmatrix}$ where $T(\vec{x}) = A\vec{x}$

Image of $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 3 & -2 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 9 \\ 7 \end{bmatrix}}$

Image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} 3 & -2 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} 3x_1 - 2x_2 \\ -5x_1 + 7x_2 \end{bmatrix}}$

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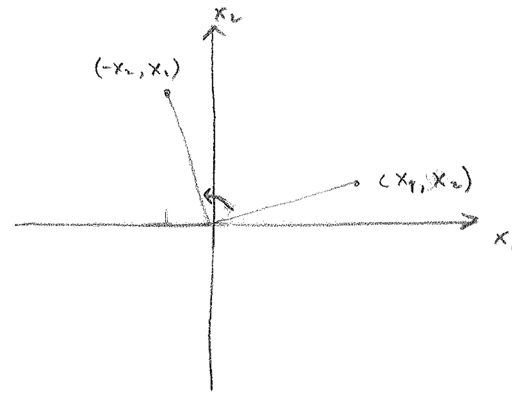
Rotation Matrices



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

perform a counterclockwise rotation by angle θ

and

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

perform a clockwise rotation by angle θ

Scaling Matrices

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s x_1 \\ t x_2 \end{bmatrix}$$

does scaling in x_1 and x_2 directions

Shear Matrices

$$\begin{bmatrix} 1 & s \\ t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + s x_2 \\ x_2 + t x_1 \end{bmatrix}$$