1.9 The Matrix of a Linear Transformation

We consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$. It turns out that one can always express this transformation as a matrix transformation, denoted $\vec{x} \mapsto A\vec{x}$.

The <u>identity Matrix</u> from $\mathbb{R}^n$ is denoted I_n and has columns $\vec{e}_i$, $i = 1, \dots, n$.

We make the observation $\vec{x} = I_n \vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$ for all $\vec{x} \in \mathbb{R}^n$.

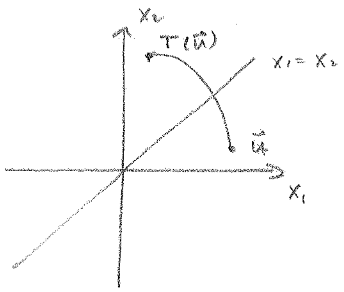
So
$$\begin{aligned}
T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\
&= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_n) && \text{since } T \text{ is linear} \\
&= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n) && \text{ditto} \\
&= [\, T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n) \,]\, \vec{x} \\
&= A\vec{x} \quad \text{where } A = [\, T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n) \,].
\end{aligned}$$

This essentially proves Thm 10, which states that
If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\exists$ a unique matrix A s.t. $T(\vec{x}) = A\vec{x}$ $\forall \vec{x} \in \mathbb{R}^n$.

We also know how to find A — its i^{th} column is just $T(\vec{e}_i)$.

In this case A is the <u>standard matrix</u> for the linear transformation T.

Ex Find the standard matrix of T if T is the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which reflects a point about the line $x_1 = x_2$



If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $T(\vec{u}) = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so

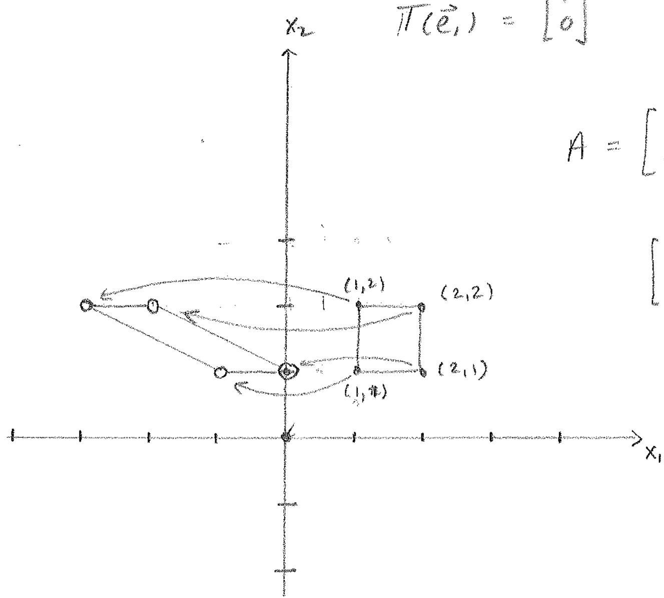
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ex Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ and

$$\begin{cases} T(\vec{e}_1) = \vec{e}_1 \\ T(\vec{e}_2) = \vec{e}_2 - 2\vec{e}_1 \end{cases}$$

Find the standard matrix of T

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ Horizontal Shear

$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_2 \end{bmatrix}$

1.9

Def

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least on $\vec{x}$ in $\mathbb{R}^n$.

If A is the standard matrix of T then saying T is onto $\mathbb{R}^m$ is equivalent to saying : given any $\vec{b}$ from $\mathbb{R}^m$, we can find an $\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$. In other words, $A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \in \mathbb{R}^m$.

Def

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$.

Again, if A is the standard matrix of T, then saying T is one-to-one is the same as saying : given any $\vec{b}$ from $\mathbb{R}^m$, $A\vec{x} = \vec{b}$ has at most one solution — it may have no solution.

Thm 11: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is 1:1 iff $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Proof

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ and for $\vec{u}, \vec{v} \in \mathbb{R}^n$ $T(\vec{u}) = \vec{a}$, $T(\vec{v}) = \vec{b}$.

Then

$$T(\vec{u}) - T(\vec{v}) = T(\vec{u} - \vec{v}) = \vec{a} - \vec{b}$$

If T is one-to-one then $\vec{a} = \vec{b}$ implies $\vec{u} = \vec{v}$ so $\vec{u} - \vec{v} = \vec{0}$ is the only solution of $T(\vec{x}) = \vec{0}$.

If $T(\vec{x}) = \vec{0}$ (i.e. $\vec{a} = \vec{b}$) has only the trivial solution then $\vec{u} = \vec{v}$. So $T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$ which says T is one-to-one.

Ex Does $T: \mathbb{R}^3 \to \mathbb{R}^3$ map $\mathbb{R}^3$ onto $\mathbb{R}^3$ if the standard matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & -5 \\ -4 & 2 & 1 \end{bmatrix} ?$$

Same question as: for any $\vec{b} \in \mathbb{R}^3$, does $A\vec{x} = \vec{b}$ have a solution? It will if it is consistent, which is ensured if A has 3 pivot columns. Since this matrix is row-equivalent to I_3 we conclude that T is onto $\mathbb{R}^3$.

Thm 12

* important!

If A is the standard matrix for a linear operator $T: \mathbb{R}^n \to \mathbb{R}^m$ then

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$

b) T is one-to-one iff the columns of A are linearly independent

Suppose $m < n$. A has more columns than rows

→ not all columns are linearly independent

→ T is not one-to-one

→ T can be onto

$m > n$ A has more rows than columns

→ not enough columns to span $\mathbb{R}^m$

→ T is not onto

→ T can be one-to-one

$m = n$: T can be both onto and one-to-one.