2.1 Matrix Operations

If A is an $m \times n$ matrix, we use a_{ij} to refer to the entry of A in the i^{th} row and the j^{th} column.

Note that a_{ij} and $\vec{a}_j$ are different - one is a scalar and one is a vector.

$A = B$ if all entries in A equal the corresponding entries in B - they must be the same size.

$C = A + B$ is defined if A and B are the same size. In this case $c_{ij} = a_{ij} + b_{ij}$.

$C = kA$ is defined for all scalars k. $c_{ij} = k a_{ij}$.

$C = A + (-1)B = A - B$: $c_{ij} = a_{ij} - b_{ij}$

Probably don't need.

Ex $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & -4 \end{bmatrix}$ $k = 2$

$$A + kB = A + 2B = \begin{bmatrix} 4 & -1 & 6 \\ 13 & 3 & -6 \end{bmatrix}$$

<u>Theorem 1</u> A, B, C are matrices of the same size and r and s are any scalars

a) $A + B = B + A$

b) $(A+B)+C = A+(B+C)$

c) $A + 0 = A$

d) $r(A+B) = rA + rB$

e) $(r+s)A = rA + sA$

f) $r(sA) = (rs)A$

Matrix Multplication

$B\vec{x}$ is a linear combination of the columns of B using the weights in $\vec{x}$. This defines a new vector.

If B is $n \times p$ and $\vec{x}$ is a p-vector, then

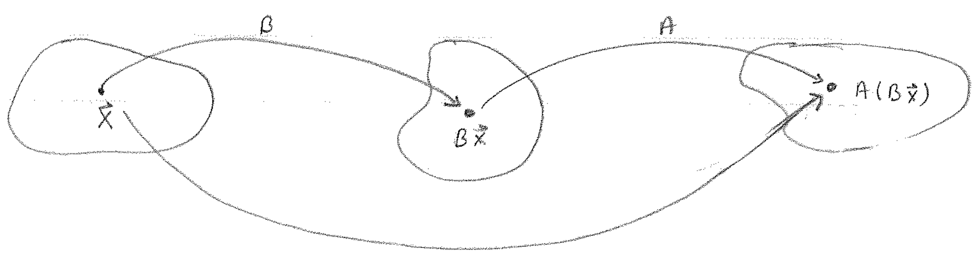
$$\vec{y} = B\vec{x} = x_1 \vec{b}_1 + x_2 \vec{b}_2 + \cdots + x_p \vec{b}_p$$

Which is a sum of n-element vectors so $\vec{y}$ itself is an n-vector.

If A is $m \times n$ then $A\vec{y}$ is defined

$$\begin{aligned} A\vec{y} &= A(x_1 \vec{b}_1 + x_2 \vec{b}_2 + \cdots + x_p \vec{b}_p) \\ &= x_1 A\vec{b}_1 + x_2 A\vec{b}_2 + \cdots + x_p A\vec{b}_p \\ &= [A\vec{b}_1 \quad A\vec{b}_2 \quad \cdots \quad A\vec{b}_p]\,\vec{x} \end{aligned}$$

Where there are p columns and m rows in this column matrix.



single transformation of $\vec{x}$ into $A(B\vec{x})$

So, we have found the standard matrix of the transformation from $\vec{x}$ to $A(B\vec{x})$ — it is $[A\vec{b}_1 \;\; A\vec{b}_2 \; \cdots \; A\vec{b}_p]$

Thus the composite mapping $A(B\vec{x})$ can be expressed as multiplication by a single matrix. We define matrix-matrix multiplication so that $A(B\vec{x}) = (AB)\vec{x}$.

Notice that if $C = AB = [A\vec{b}_1 \;\; A\vec{b}_2 \; \cdots \; A\vec{b}_p]$ that the j^{th} column of C is given by $\vec{c}_j = A\vec{b}_j$.

$$\vec{c}_j = A\vec{b}_j = b_{1j}\vec{a}_1 + b_{2j}\vec{a}_2 + \cdots + b_{nj}\vec{a}_n$$

So, what is c_{ij}? $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Which leads to the familiar "row-by-column" approach to matrix-matrix multiplication.

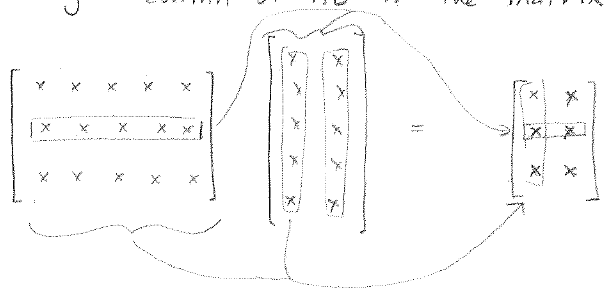
Easy to determine size of product

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} = \underset{m \times p}{C}$$

these must match

Observe:

1) the i^{th} row of AB is the i^{th} row of A times the matrix B
2) the j^{th} column of AB is the matrix A times the j^{th} column of B



Thm 2

Properties: A is $m \times n$, B & C are appropriate sizes for indicated operation

a) $A(BC) = (AB)C$ assoc. for multiplication

b) $A(B+C) = AB + AC$ distributive

c) $(A+B)C = AC + BC$

d) $r(AB) = (rA)B = A(rB)$

e) $I_m A = A = A I_n$ often I is used without a subscript

Proof of b

j^{th} column of $B+C$ is $\vec{b}_j + \vec{c}_j$. The j^{th} column of the product $A(B+C)$ is $A(\vec{b}_j + \vec{c}_j) = A\vec{b}_j + A\vec{c}_j$.
However $A\vec{b}_j$ is the j^{th} column of AB & $A\vec{c}_j$ is the j^{th} column of AC. Since this is true of all columns we conclude

$\therefore A(B+C) = AB + AC$

The transpose of a matrix is obtained by exchanging the rows and columns. If A is an $m \times n$ matrix, A^T is an $n \times m$ matrix such that the ij element of the transpose is a_{ji}

Octave: A'

If $B = A^T$ then $b_{ij} = a_{ji}$

Ex $A = \begin{bmatrix} 2 & 6 & 9 & 0 \\ 1 & 3 & 4 & -1 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 1 \\ 6 & 3 \\ 9 & 4 \\ 0 & -1 \end{bmatrix}$

Thm 3

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(rA)^T = rA^T \quad \forall r$

d) $(AB)^T = B^T A^T$ notice the order of A and B changes.

We define A^k for nonnegative integers k to be the matrix A times itself k times. If $k=0$ then $A = I$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}} I$$

notice that A must be $n \times n$, i.e. square.

Finally, we note that while some aspects of matrix-matrix multiplication are similar to scalar multiplication, some things are quite different.

In general matrix multiplication is not commonunitive, $AB \neq BA$. In fact, BA may not even be defined: if A is $m \times n$ and B is $n \times p$ then AB is defined but BA is defined only if $m = p$.

Ex $\quad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 4 & 3 \\ 1 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 2 & 7 \\ 1 & 2 \end{bmatrix}$$

$$AB \neq BA$$

If $AB = AC$, what can we conclude about B and C?
$\rightarrow$ they have the same dimensions.
This does not imply that $B = C$.

If $AB = 0$, does this mean that either A or B is zero — no.

For example find 2 vectors $\vec{x}_1$ and $\vec{x}_2$ for which are nontrivial solutions of $A\vec{x}_i = \vec{0}$ and let these be the columns of B. Then $AB = 0$ but neither A or B is zero.

Ex $\quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$