2.3 Characterizations of Invertible Matrices

Thm 8 Let A be a square $n \times n$ matrix. The following statements are equivalent:

a) A is an invertible matrix
b) A is row equivalent to I_n
c) A has n pivot positions
d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution
e) The columns of A form a linearly independent set
f) The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.
g) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$

[can also be stated as "$A\vec{x} = \vec{b}$ has a <u>unique</u> solution for each $\vec{b} \in \mathbb{R}^n$"].

h) The columns of A span $\mathbb{R}^n$
i) The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
j) $\exists$ an $n \times n$ matrix C s.t. $CA = I$
k) $\exists$ an $n \times n$ matrix D s.t. $AD = I$
l) A^T is an invertible matrix.

The set of $n \times n$ matrices is partitioned into two disjoint sets.

invertible (nonsingular)	non-invertible (Singular)

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if
$\exists$ $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$S(T(\vec{x})) = \vec{x} \qquad \forall \vec{x} \in \mathbb{R}^n$$
$$T(S(\vec{x})) = \vec{x} \qquad \forall \vec{x} \in \mathbb{R}^n$$

Thm 9

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix of that transformation. Then T is invertible iff A is an invertible matrix.

Ex. #11. a) T $d \Rightarrow b$ b) T $h \Rightarrow e$ c) F d) T $\neg d \Rightarrow \neg c$ e) T $\neg l \Rightarrow \neg a$

Ex #13. $\begin{bmatrix} a & x_1 & x_2 \\ 0 & b & x_3 \\ 0 & 0 & c \end{bmatrix}$ If $a, b, c \neq 0$ we can row reduce this to I_3

What if a, b or $c = 0$?

$a = 0 \Rightarrow$ column of zeros, cannot row reduce to I

$b = 0 \Rightarrow$ only 2 pivot columns

$c = 0 \Rightarrow$ ditto

A triangular $n \times n$ matrix is invertible iff each of its diagonal entries is nonzero.