2.4 Partitioned Matrices

Given an $m \times n$ matrix, we can subdivide or partition it into smaller blocks, called blocks or submatrices.

Ex

$$A = \left[\begin{array}{ccc|cc} 2 & 1 & 6 & 8 & 9 \\ \hline 1 & 3 & 0 & 0 & 2 \\ -1 & 6 & 2 & 1 & 5 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

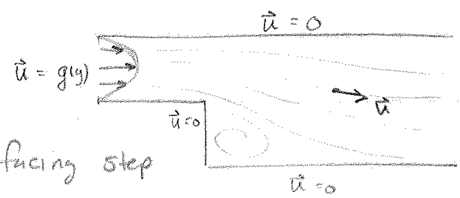
$$A_{11} = \begin{bmatrix} 2 & 1 & 6 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 8 & 9 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 6 & 2 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$$

Partitioned matrices are particularly useful in numerical computation as they provide a way to work with only part of a larger matrix system, thus allowing systems to big to fit into memory at once to be worked with.

Often, matrix partition are determined by some physical aspect of the problem being solved.

Ex Stokes Flow over a backward facing step



$\vec{u}$ velocity

p pressure

yields

$$\begin{bmatrix} K_x & 0 & B_x \\ 0 & K_y & B_y \\ B_x^T & B_y^T & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f}_x \\ \vec{f}_y \\ \vec{0} \end{bmatrix}$$

where $\vec{u}_x$ = array of x-components of velocity

$\vec{f}_x$ = array of x-components of any driving force (gravity?)

This system can be expressed more compactly as

$$\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ 0 \end{bmatrix}$$

What is meant by the product on the left-hand-side and how do we compute it? We can actually use the row-column rule - but must remember that matrix multiplication is not communitive. So,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Of course, A and B must each have an appropriate number of blocks of the appropriate sizes. If this is true for A and B we say that A and B are <u>conformable</u> for block multiplication

Actually, we've been using a particular partitioning scheme repeatedly - namely $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$ where we think of each column of A as an $m \times 1$ matrix.

Note that A & B do not need the same number of submatrices, only that the correct number of correctly dimensioned submatrices exist.

Ex $\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{0} \end{bmatrix} \rightarrow \left.\begin{aligned} K\vec{u} + B\vec{p} &= \vec{f} \\ B^T\vec{u} &= \vec{0} \end{aligned}\right\}$ set of 2 matrix equations

$B^T\vec{u} = \vec{0}$ has nontrivial solutions

$\rightarrow$ columns of B^T linearly dependent

Consider A $(m \times n)$ and B $(n \times p)$ so that AB is defined.

One conformable partition of AB would be to make each column of A a block and each row of B a block.

$$AB = [\,col_1(A) \quad col_2(A) \cdots col_n(A)\,] \begin{bmatrix} row_1(B) \\ row_2(B) \\ \vdots \\ row_n(B) \end{bmatrix}$$

$$= \sum_{i=1}^{n} col_i(A)\, row_i(B)$$

The product $col_i(A)\, row_i(B)$ is called an <u>outer product</u>. What is the size of the result?

$col_i(A)$ is a $m \times 1$ matrix

$row_i(B)$ is a $1 \times p$ matrix

So the product will be $m \times p$, as is AB.

Inverses of Partitioned Matrices

Invert $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. If A is invertible then A_{11} and A_{22} will be invertible - why?

Find B s.t. $AB = BA = I$. Text does this. We try another approach

$$\left[\begin{array}{cc|cc} A_{11} & A_{12} & I & 0 \\ 0 & A_{22} & 0 & I \end{array}\right] \sim \left[\begin{array}{cc|cc} A_{11} & A_{12} & I & 0 \\ 0 & I & 0 & A_{22}^{-1} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} A_{11} & 0 & I & -A_{12}A_{22}^{-1} \\ 0 & I & 0 & A_{22}^{-1} \end{array}\right] \sim \left[\begin{array}{cc|cc} I & 0 & A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & I & 0 & A_{22}^{-1} \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$

$\Rightarrow$ Notice that this method only works if we mult. only on left. <u>why?</u>

(approach based on multiple systems of form $A\vec{x} = \vec{e}_i$)

<u>Ex</u> $\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{f} \\ \vec{0} \end{bmatrix} \rightarrow \begin{array}{l} K\vec{u} + B\vec{p} = \vec{f} \\ B^T\vec{u} = \vec{0} \end{array}$

we can solve this for $\vec{u}$ and $\vec{p}$ by: ① solving for $\vec{u}$ in the first eq.; ② putting this $\vec{u}$ into the second; ③ solve for $\vec{p}$ explicitly then in turn ④ find $\vec{u}$.

$$\vec{u} = K^{-1}(\vec{f} - B\vec{p})$$

$$\text{so} \quad B^T\vec{u} = B^T K^{-1}(\vec{f} - B\vec{p}) = 0$$

$$B^T K^{-1} \vec{f} = B^T K^{-1} B \vec{p}$$

so $\vec{p} = (B^T K^{-1} B)^{-1} B^T K^{-1} \vec{f}$

once $\vec{p}$ is known, $\vec{u}$ is given by $\vec{u} = K^{-1}\vec{f} - K^{-1}B\vec{p}$.

In class (on board?)

\# 9, 11, 14