2.6 The Leontief Input-Output Model

Consider an economic system composed of 2 industries: coal and steel. Suppose that

1. To produce, \$1 worth of output, the coal industry requires \$0.10 of its own product and \$0.20 of steel.
2. To produce \$1 worth of output, the steel industry requires \$0.10 of its own product and \$0.80 of coal.

Inputs consumed per unit output

Purchased from	coal	steel
coal	0.10	0.80
steel	0.20	0.10

These columns form "unit consumption vectors".

Suppose further that elements outside the coal and steel industries demand \$10,000 worth of coal and \$20,000 worth of steel.

If c and s are the amounts of coal and steel which must be produced, then

$$c = 0.10c + 0.80s + 10,000$$

$$s = \underbrace{0.20c + 0.10s}_{\substack{\text{intermediate demand} \\ \text{(internal demand)}}} + \underbrace{20,000}_{\substack{\text{Final Demand} \\ \text{(external demand)}}}$$

We can write this in matrix form

$$\begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} 0.10 & 0.80 \\ 0.20 & 0.10 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} + \begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$$

or

$$\underset{\text{amount produced}}{\vec{x}} = \underbrace{C\vec{x}}_{\text{intermediate demand}} + \underset{\text{final demand}}{\vec{d}}$$

C is the "consumption matrix"

To solve this

$$\vec{x} - C\vec{x} = \vec{d}$$

$$I\vec{x} - C\vec{x} = \vec{d}$$

$$\underbrace{(I - C)}_{A}\,\vec{x} = \underbrace{\vec{d}}_{\vec{b}}$$

So we have $A\vec{x} = \vec{b}$

For our problem

$$I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.10 & 0.80 \\ 0.20 & 0.10 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.8 \\ -0.2 & 0.9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0.9 & -0.8 & 10000 \\ -0.2 & 0.9 & 20000 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 38461.54 \\ 0 & 1 & 30769.23 \end{array}\right]$$

So to meet the final and intermediate demand, Coal must produce \$38,461.54 worth of output and Steel must produce \$30,769.23 worth of output.

The eq. $\vec{x} = C\vec{x} + \vec{d}$ comes from a physical problem that would not have negative numbers in the solution so we expect non-negative entries in $\vec{x}$. We also expect a unique solution, which would imply that $I-C$ is invertible.

Thm 11

Let the $n \times n$ matrix C and the n-vector $\vec{d}$ have only nonnegative entries. If the sum of the elements in each column of C is less than 1 then $(I-C)^{-1}$ exists and $\vec{x} = (I-C)^{-1}\vec{d}$ is a unique solution with nonnegative entries.

Column sum of $C < 1$:

don't want any sector to consume more dollar value of products than it produces.

$\vec{x}$ has nonnegative entries :

want all production levels to be positive, or at least 0.

Notice how the physical constraints show up mathematically.

Now we consider $(I-C)^{-1}$ in more detail. We required in Thm 11 that the column sum of each column of C be strictly less than one. It can be shown that if this is true then $C^m \to 0$ as $m \to \infty$, sort of like $a^m \to 0$ as $m \to \infty$ if $|a| < 1$.

Notice that

$$(I-C)(I+C+C^2+\cdots+C^{m-1})$$
$$= \underbrace{I+C+C^2+\cdots+C^{m-1}}_{I\cdot(\ )} \underbrace{-C-C^2-C^3-\cdots-C^m}_{-C(\ )}$$
$$= I + C - C + C^2 - C^2 + \cdots + C^{m-1} - C^{m-1} - C^m$$
$$= I - C^m.$$

This means $I + C + C^2 + \cdots + C^{m-1} = (I-C)^{-1}(I-C^m)$

If $C^m \to 0$ as $m \to \infty$ then

$$(I-C)^{-1} = I + C + C^2 + C^3 + \cdots$$

If C^m goes to zero quickly as m increases then we can use $I + C + C^2 + \cdots + C^{m-1}$ as an approximation for $(I-C)^{-1}$.

Now

$$\vec{x} = (I-C)^{-1}\vec{d} = (I + C + C^2 + \cdots)\vec{d}$$
$$\vec{x} = \underbrace{\vec{d}}_{\text{final demand}} + \underbrace{C\vec{d}}_{\text{first intermediate demand}} + \underbrace{C^2\vec{d}}_{\text{second intermediate demand}} + \underbrace{C^3\vec{d}}_{\text{third intermediate demand}} + \cdots$$

Ex. If $D_m = I + C + C^2 + \cdots + C^m$ then we can generate D_m using the recurrence relation $D_{m+1} = I + CD_m$ with $D_0 = I$.

$D_1 = I + CD_0 = I + C$, $D_2 = I + C(I+C) = I + C + C^2$, $D_3 = I + C(I + C + C^2)$

If $C = \begin{bmatrix} 0.1 & 0.8 \\ 0.2 & 0.1 \end{bmatrix}$ then $D_{10} = \begin{bmatrix} 1.38413 & 1.22979 \\ 0.30745 & 1.38413 \end{bmatrix}$

note that $(I-C)^{-1} = \begin{bmatrix} 1.38462 & 1.23077 \\ 0.30769 & 1.38462 \end{bmatrix}$