3.1 Introduction To Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then,

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We compute determinants only for square matrices.

For a 3×3 matrix, the work is more complicated

<u>Ex</u> $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

If we use A_{ij} to mean the $(n-1) \times (n-1)$ matrix which results if the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column of A are removed, then we can say

$$\det A = \sum_{j=1}^{3} (-1)^{j+1} a_{1j} \det A_{1j} = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

The A_{ij} are 2×2 matrices and so the determinants of them are straight forward.

We can extend this in a useful way to matrices of higher dimension.

For $n \geq 2$ the <u>determinant</u> of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{for any } i : 1 \leq i \leq n$$

or

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{for any } j : 1 \leq j \leq n$$

This is called <u>Cofactor expansion</u>, and is a recursive definition since to find the det. of an $n \times n$ matrix, one must find n determinants of $(n-1) \times (n-1)$ matrices.

<u>Ex</u> $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -4 & 1 & 3 \end{bmatrix}$

$$\begin{aligned} \det A &= 1\,(1\cdot 3 - 1\cdot 3) - 0\,(2\cdot 3 - (-4)3) + (-1)\,(2\cdot 1 - (-4)\,1) \\ &= 0 - 0 - (2+4) = -6 \end{aligned}$$

Note that we can expand about any row or column, so it makes sense to use the row or column which has the largest number of zeros.

<u>Ex</u> $A = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ $\quad \det A = 5\,(2\cdot 0 - 1\cdot 3) - 0\,(\quad) + 0\,(\quad)$

$= -15$

Just be sure to get the signs correct!

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

If A is a <u>diagonal matrix</u> $A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$

$$\det A = a_{11} \det A_{11} = a_{11} a_{22} \det (A_{11})_{11} = \cdots = a_{11} a_{22} \cdots a_{nn}$$

So

The determinant of a diagonal matrix is the product of the entries on the diagonal

If A is a triangular matrix (upper or lower) we find the same result

$$A = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$$\det A = a_{11} \det A_{11} = a_{11} a_{22} \det (A_{11})_{11} = \cdots = a_{11} a_{22} \cdots a_{nn}$$

The determinant of a triangular matrix is the product of the entries on the diagonal.