

3.2 Properties of Determinants

Consider the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. How do row operations on A effect the determinant?

① Row replacement: $kR_1 + R_2 \rightarrow R_2$ gives $\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix} &= a(d+kb) - b(c+ka) \\ &= ad + kab - bc - kab \\ &= ad - bc \\ &= \det A \end{aligned}$$

This row replacement did not change the value of the determinant. In fact this is the case for any row replacement on any size matrix.

Row Replacement operations do not change the determinant of a matrix

proof in text

② Row Scaling: $kR_1 \rightarrow R_1$ gives $\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$

$$\begin{aligned} \det \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} &= kad - kbc \\ &= k(ad - bc) \\ &= k \det A \end{aligned}$$

This is easy to prove for any row of any size

Scaling a row by a factor k results in a matrix whose determinant is k times the determinant of the original matrix

matrix using cofactor exp about the scaled row.

3.2 Properties of Determinants

2

③ Row Swapping: $R_1 \leftrightarrow R_2$ $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{aligned}\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} &= cb - ad \\ &= -(ad - cb) \\ &= -\det A.\end{aligned}$$

Swapping two rows of a matrix changes the sign of the determinant.

Recall that the determinant of a triangular matrix is the product of its diagonal elements. Idea: We should be able to perform row replacement and row swapping operations on any square matrix to obtain a triangular matrix, and because row replacement does not change the determinant and row swapping only changes the sign of the determinant, we can compute the determinant of a square matrix by finding the corresponding triangular matrix, find its determinant, and multiply this by $(-1)^r$ where r is the number of row swaps.

If $A \sim U$ and U is obtained from A by row replacements and r row swaps, then

$$\det A = (-1)^r \det U$$

3.2 Properties of Determinants

3

Ex $A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -12 & 2 \\ -1 & 7 & 0 \end{bmatrix}$

$$A \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 0 & -4 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{\text{row exchange}} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & 1 \\ 0 & 0 & -4 \end{bmatrix} = U \quad \det U = (3)(5)(-4) = -60$$

1 row exchange so

$$\det A = (-1)^1 (-60) = 60$$

$$\boxed{\det A = 60}$$

Note: If $A = LU$ then

- 1) $|\det A| = |\det U|$
- 2) $\det A = 0$ iff $\det U = 0$
- 3) $\det U = 0$ iff there is a zero on the diagonal
- 4) U is invertible iff it has n pivots
- 5) U is invertible iff it has no zeros on the diagonal
- 6) $A \sim I$ iff $U \sim I$
- 7) A is invertible iff U is invertible

$\therefore$ $\boxed{\text{The } nxn \text{ matrix } A \text{ is invertible iff } \det A \neq 0}$

3.2 Properties of Determinants

4

Some other interesting properties

Q. What is the relation between $\det A$ and $\det kA$?

A. In the matrix kA each row of A has been scaled by k . Since scaling one row of A changes the determinant by a factor of k , scaling n rows of A will change the determinant by a factor of k^n .

$$\det kA = k^n \det A \text{ if } A \text{ is } n \times n$$

Q. What is $\det A^T$?

A. Suppose we compute $\det A$ using cofactor expansion about row i . The identical computations will be performed when computing $\det A^T$ using column i . Thus

$$\det A = \det A^T$$

Distributive Property: $\det(AB) = \det A \cdot \det B$

Proof: If $\det A = 0$ then A is not invertible. This tells us that AB is not invertible [If AB is invertible then $\exists$ an $n \times n$ matrix W s.t. $ABW = I$ but then $A(BW) = I$ so A is invertible]. If AB is not invertible then $\det AB = 0$ and our statement is true.

3.2 Properties of Determinants

5

If A is invertible then $A \sim I$ and so there exist elementary matrices $E_p, E_{p-1}, \dots, E_1$ such that

$$A = E_p E_{p-1} \dots E_1 \cdot I = E_p E_{p-1} \dots E_1$$

$$\begin{aligned} \det(AB) &= \det(E_p E_{p-1} \dots E_1 B) \\ &= \det(E_p) \det(E_{p-1} \dots E_1 B) \end{aligned}$$

because $\det(E_p)$ will be 1, -1, or k

1 if E_p is a row replacement

-1 if E_p is a row swap

k if E_p scales a row by k

Continuing we have

$$\begin{aligned} \det(AB) &= \det(E_p) \det(E_{p-1}) \dots \det(E_1) \det(B) \\ &= \det(E_p E_{p-1} \dots E_1) \det(B) \\ &= \det A \cdot \det B. \end{aligned}$$