3.3 Cramers Rule, Volume & Linear Transformations

We can use the determinant (at least in theory) to compute the solution of a linear system using a rule known as Cramer's Rule.

First, we let $A_i(\vec{b})$ be the matrix obtained by replacing the i^{th} column of A by the vector $\vec{b}$.

Cramer's Rule:

Let A be an invertible $n \times n$ matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A} \qquad i = 1, 2, \dots, n$$

Proof: use $|A|$ to denote $\det A$

$$\begin{aligned}
|A \cdot I_i(\vec{x})| &= |A \cdot [\vec{e}_1 \ \vec{e}_2 \cdots \vec{x} \cdots \vec{e}_n]| \\
&= |[A\vec{e}_1 \ A\vec{e}_2 \cdots A\vec{x} \cdots A\vec{e}_n]| \\
&= |[\vec{a}_1 \ \vec{a}_2 \cdots \vec{b} \cdots \vec{a}_n]| \\
&= |A_i(\vec{b})| \\
|A|\,|I_i(\vec{x})| &= |A_i(\vec{b})| \\
|A| \ x_i &= |A_i(\vec{b})|
\end{aligned}$$

$$\rightarrow \quad x_i = \frac{\det A_i(\vec{b})}{\det A} \quad \checkmark$$

Ex For what values of s does the system

$$sx + 4y = 1$$
$$-x - 2sy = 3$$

have a unique solution? In this case, find the solution.

We will need $\det A = \det \begin{bmatrix} s & 4 \\ -1 & -2s \end{bmatrix} = -2s^2 + 4 = 4 - 2s^2 \neq 0$

So we need

$$2 \neq s^2 \rightarrow s \neq \pm\sqrt{2}$$

So, we will have a unique solution if $s \neq \pm\sqrt{2}$

$$x = \frac{\det \begin{bmatrix} 1 & 4 \\ 3 & -2s \end{bmatrix}}{4 - 2s^2} = \frac{-2s - 12}{4 - 2s^2} = \boxed{-\frac{s+6}{2-s^2}}$$

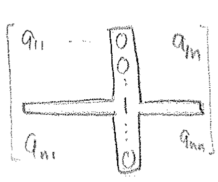
$$y = \frac{\det \begin{bmatrix} s & 1 \\ -1 & 3 \end{bmatrix}}{4 - 2s^2} = \boxed{\frac{3s+1}{4-2s^2}}$$

Inverses

We know that one way to find the inverse of a matrix is to solve $A\vec{x}_j = \vec{e}_j$ for $j = 1, \dots, n$. Then the $\vec{x}_j$'s form the columns of A^{-1}! Thus, the i^{th} entry of $\vec{x}_j$ is the (ij)-entry of A^{-1}

Using Cramer's Rule, we know $x_i = \dfrac{\det A_i(\vec{e}_j)}{\det A}$

Notice that $\det A_i(\vec{e}_j) = (-1)^{i+j} \det A_{ji}$



A with the j^{th} row and i^{th} column removed.

We will let $C_{ji} = (-1)^{i+j} \det A_{ji}$ [or $C_{ij} = (-1)^{i+j} \det A_{ij}$]

Thus $A^{-1}_{ij} = x_i = \dfrac{\det A_i(\vec{e}_j)}{\det A} = \dfrac{C_{ji}}{\det A}$

So

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & \ddots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

note that indicies are reversed from normal.

The matrix of the cofactors is called the adjugate of A

$\operatorname{adj} A$

So $A^{-1} = \dfrac{1}{\det A} \operatorname{adj} A$

Ex Find A^{-1} if $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix}$

$\det A = 1(1\cdot 0 - (-1)(1)) - 0 + 3(2(-1) - 4\cdot 1) = 1 - 18 = -17$

$C_{11} = +\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$ $\quad C_{12} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3$ $\quad C_{13} = +\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3$

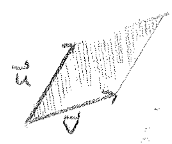
$C_{21} = -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 4$ $\quad C_{22} = +\begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} = -12$ $\quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 5$

$C_{31} = +\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} = -6$ $\quad C_{32} = -\begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix} = 1$ $\quad C_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$

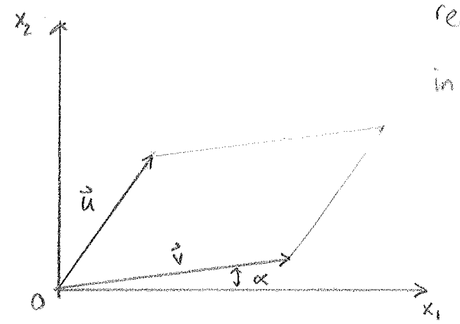
So $A^{-1} = \dfrac{1}{-17}\begin{bmatrix} 1 & 4 & -6 \\ -3 & -12 & 1 \\ -3 & 5 & 1 \end{bmatrix}$

Area & Volume

In $\mathbb{R}^2$ two nonparallel vectors define a parallelogram.



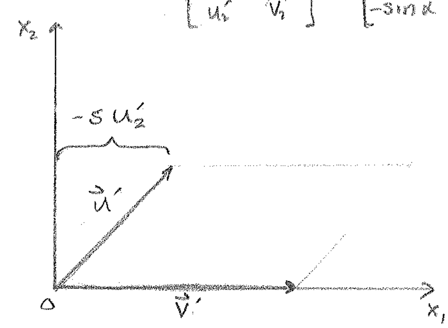
Now we consider the operations necessary to transform this into a rectangle aligned with the coordinate axes in $\mathbb{R}^2$



we first rotate both vectors by α to line $\vec{v}$ up with the x_1-axis

$$\begin{bmatrix} u_1' & v_1' \\ u_2' & v_2' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

Note: $v_2' = 0$

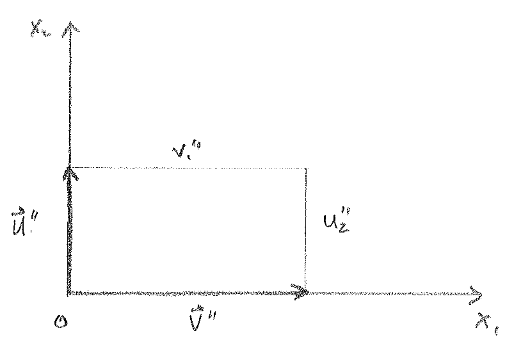


Next shear to align $\vec{u}'$ with x_2-axis. Need

$$0 = u_1'' = u_1' + s u_2'$$

$$\begin{bmatrix} u_1'' & v_1'' \\ u_2'' & v_2'' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1' & v_1' \\ u_2' & v_2' \end{bmatrix}$$

Note $u_1'' = 0$, $v_2'' = 0$



Area $= (v_1'')(u_2'')$ but this is just

$$\det \begin{bmatrix} u_1'' & v_1'' \\ u_2'' & v_2'' \end{bmatrix} = \det \begin{bmatrix} 0 & v_1'' \\ u_2'' & 0 \end{bmatrix} = -(v_1'' u_2'')$$

So

$$\text{Area} = \left| \det [\vec{u}'' \ \vec{v}''] \right|$$

So, how does this compare with original?

Notice that neither rotating nor shearing changed the area.

$$\det\begin{bmatrix} u_1'' & v_1'' \\ u_2'' & v_2'' \end{bmatrix} = \det\left(\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot \det\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

$$= 1 \cdot 1 \cdot \det\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

So

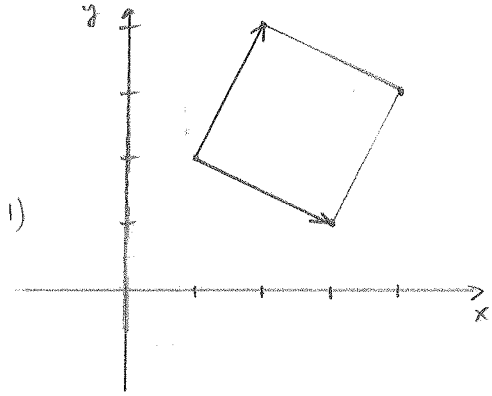
Area of parallelogram formed by $\vec{u}$ and $\vec{v}$ is $|\det[\vec{u}\ \vec{v}]|$

Same works for volumes in $\mathbb{R}^3$..., volumes of parallelepipeds

S. <u>Thm</u>

If A is 2×2, the area of the parallelogram determined by the columns of A is $|\det A|$.
If A is 3×3, the volume of of the parallelepiped determined by the columns of A is $|\det A|$.

<u>Ex</u> Find the area of the parallelogram determined by $(1,2)$, $(3,1)$, $(2,4)$, $(4,3)$



① Translate to origin

$(0,0)$, $(2,-1)$, $(1,2)$, $(3,1)$

② form matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

③ find area Area $= \det A = |2\cdot 2 - (-1)(1)| = |4+1| = 5$

<u>Transformations</u>

If $T:\mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix A (2×2), and if S is a parallelogram in $\mathbb{R}^2$ then

$$\{\text{area } T(S)\} = |\det A| \cdot \{\text{area } S\}$$

If $T:\mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix A (3×3) and if S is a parallelepiped in $\mathbb{R}^3$ then

$$\{\text{Volume } T(S)\} = |\det A| \cdot \{\text{Volumn } S\}$$

We already saw this twice - in both cases $\det A = 1$

① Shear matrix $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 + su_2 & v_1 + sv_2 \\ u_2 & v_2 \end{bmatrix}$

$$\begin{vmatrix} u_1 + su_2 & v_1 + sv_2 \\ u_2 & v_2 \end{vmatrix} = (u_1 + su_2)v_2 - (v_1 + sv_2)u_2 = u_1v_2 + su_2v_2 - v_1u_2 - su_2v_2$$
$$= u_1v_2 - v_1u_2$$

$$\left\{\text{area } \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\right\} = \left|\det \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\right| \cdot \{\text{area } [\vec{u}\ \vec{v}]\}$$
$$= 1 \cdot \{\text{area } [\vec{u}\ \vec{v}]\}$$

② Rotation matrix $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} [\vec{u}\ \vec{v}] = [\vec{u}'\ \vec{v}']$

$$\{\text{area } [\vec{u}'\ \vec{v}']\} = \left|\det \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\right| \cdot \{\text{area } [\vec{u}\ \vec{v}]\}$$
$$= 1 \cdot \{\text{area } [\vec{u}\ \vec{v}]\}$$