4.1 Vector Spaces and Subspaces

Def A vector space is a nonempty set V of vectors on which are defined two operations, addition and multiplication by scalars, subject to the ten axioms

1. $\vec{u} + \vec{v} \in V$ [Closure under addition]
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. $\exists\ \vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u}$
5. $\forall\ \vec{u} \in V$ $\exists$ a vector $-\vec{u} \in V$ s.t $\vec{u} + (-\vec{u}) = \vec{0}$
6. $c\vec{u} \in V$ [Closure under scalar multiplication]
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars c and d.

- All the spaces $\mathbb{R}^n$, $n \geq 1$, are vector spaces.

Ex Let V be the set of all 2^{nd} order linear polynomials: $\mathbb{P}_2$

$$p(t) = a_0 + a_1 t + a_2 t^2$$

If $\vec{u} = u_0 + u_1 t + u_2 t^2$

$\vec{v} = v_0 + v_1 t + v_2 t^2$

① $\vec{u} + \vec{v} = u_0 + u_1 t + u_2 t^2 + v_0 + v_1 t + v_2 t^2$

$= (u_0 + v_0) + (u_1 + v_1)t + (u_2 + v_2)t^2 \in V$

②, ③ follow.

④ $\vec{0} = 0 + 0t + 0t^2$ ✓

⑤ $-\vec{u} = -u_0 + (-u_1)t + (-u_2)t^2$ ✓

⑥ - ⑩ ✓

In fact, if $\mathbb{P}_n$ is the set of all polynomials of order n (or less) then $\mathbb{P}_n$ is a Vector Space.

Ex Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : -1 \le x \le 1, \ -1 \le y \le 1 \right\}$. Is V a vector space?

To be a vector space, we must have <u>closure</u> under addition (axiom 1). Can we think of an example which violates this axiom?

$$\vec{u} = \begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} \qquad \text{Both } \vec{u}, \vec{v} \in V$$

$$\vec{u} + \vec{v} = \begin{bmatrix} 1.1 \\ 1.1 \end{bmatrix}$$ which is not in V.

$\boxed{V \text{ is not a vector space}}$

In many cases, subsets of a vector space are themselves vector spaces. These "Subspaces" are vector spaces contained within other vector spaces.

Ex If $\mathbb{P}$ is the set of all polynomials with real coefficients then $\mathbb{P}_2$ is a subspace of $\mathbb{P}$ since $\mathbb{P}_2$ is a vector space and a subset of $\mathbb{P}$.

Note that $\mathbb{R}^3 \not\subset \mathbb{R}^4$ since no vector in $\mathbb{R}^3$ is in $\mathbb{R}^4$

4.1

A subspace of a vector space V is a subset H of V such that H itself is a vector space under the same operations that are defined on V.

$\{\vec{0}\}$ This is a set consisting only of the zero vector. It is a subspace of the set V, in which it is the zero vector.

Ex $H = \left\{ \begin{bmatrix} s \\ 2+s \end{bmatrix} : s \in \mathbb{R} \right\}$ Is H a subspace of $\mathbb{R}^2$?

No - $\vec{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \notin H$.

Ex $H = \left\{ \begin{bmatrix} s \\ 2s \end{bmatrix} : s \in \mathbb{R} \right\}$ Is H a subspace of $\mathbb{R}^2$?

$\vec{u} = \begin{bmatrix} s \\ 2s \end{bmatrix}$ $\vec{v} = \begin{bmatrix} t \\ 2t \end{bmatrix}$ $\vec{u}+\vec{v} = \begin{bmatrix} s+t \\ 2(s+t) \end{bmatrix} \in V$ (axiom 1 holds)

2,3 follow. 4: $s=0$, 5: $-\vec{u} = \begin{bmatrix} -s \\ -2s \end{bmatrix}$, 6-10 ok.

Yes, H is a subspace of $\mathbb{R}^2$.

Thm A subset H of a vector space V is a subspace iff the following conditions are all satisfied:

a) The zero vector of V is in H.

b) If $\vec{u}, \vec{v} \in H$, then $\vec{u}+\vec{v} \in H$.

c) If $\vec{u} \in H$ then $c\vec{u} \in H$ for any scalar c.

Need zero vector from V, closure under addition, and closure under scalar multiplication

A vector space V is spanned by a set of vectors $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ if every vector in V is a linear combination of the vectors in S.

<u>Ex</u> Given $\vec{v}_1, \vec{v}_2 \in V$, let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Show that H is a subspace of V.

a) The zero vector of V is in H since $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$

b) $\vec{x}_1 = a_1\vec{v}_1 + a_2\vec{v}_2$, $\vec{x}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$ so $\vec{x}_1, \vec{x}_2$ are any vectors in H. Then $\vec{x}_1 + \vec{x}_2 = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 \in H$

c) $c\vec{x}_1 = (ca_1)\vec{v}_1 + (ca_2)\vec{v}_2 \in H$

<u>Ex</u> If $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, what must be true about $\vec{v}_3$?

$$\exists\ c_1, c_2 \text{ s.t. } \vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$$

Thm: If $\vec{v}_1, \ldots, \vec{v}_p$ are in a vector space V, then $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of V.

Do in class 9, 11, 20, 23 a-d
(HW includes 10, 12, 24)