4.2 Null Spaces, Column Spaces & Linear Transformations

Def Given an $m \times n$ matrix A, the null space of A is the set Nul A of all solutions to the homogeneous eq. $A\vec{x} = \vec{0}$.

$$\text{Nul } A = \{\vec{x} : \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\}$$

Ex $A = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ does $\vec{v} = \begin{bmatrix} 0 \\ -8 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ belong to Nul A?

$$A\vec{v} = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -8 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $\boxed{\text{YES}}$

Is the null space really a vector space? (subspace of $\mathbb{R}^n$)

① Does it have the $\vec{0}$ element from $\mathbb{R}^n$? $A\vec{0} = \vec{0}$ ✓ yes

② Does $A(\vec{u} + \vec{v}) = \vec{0}$ if $\vec{u}, \vec{v} \in$ Nul A?

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$ yes

③ Does $A(c\vec{u}) = \vec{0}$ if $\vec{u} \in$ Nul A

$$A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$$ yes

Thm If A is an $m \times n$ matrix then Nul A is a subspace of $\mathbb{R}^n$

How do we find the null space of a matrix?

Recall section 1.5 and consider $A\vec{x}=\vec{0}$ with

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$ as before. Then the linear system $A\vec{x}=\vec{0}$ is

$$\begin{aligned} x_1 \quad - x_3 \quad + 3x_5 &= 0 \\ x_2 + 2x_3 \quad + 2x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

Since A is already in reduced echelon form.

So

$$\begin{aligned} x_1 &= x_3 - 3x_5 \\ x_2 &= -2x_3 - 2x_5 \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= x_5 \\ x_5 &= x_5 \quad \text{(free)} \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_5 \\ -2x_3 - 2x_5 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, every vector $\vec{x}$ which is in Nul A is a linear combination of $\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Thus we can say $\vec{u}_1$ and $\vec{u}_2$ <u>span</u> Nul A, or they form a <u>spanning set</u> of the null space.

$$\text{Nul } A = \text{span}\{\vec{u}_1, \vec{u}_2\}$$

Def Given an $m \times n$ matrix A, the <u>column space</u> of A is the set $\text{Col } A$ of all linear combinations of the columns of A.

$$\text{Col } A = \{ \vec{b} : \vec{b} \in \mathbb{R}^m \text{ and } \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n \}$$

Thm If A is an $m \times n$ matrix then $\text{Col } A$ is a subspace of $\mathbb{R}^m$.

This is not necessary since we know that if $\vec{v}_1, \dots, \vec{v}_p$ are vectors in a vector space V then $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V.

Proof: $\vec{b} = A\vec{0} = \vec{0}$ is a zero element in both $\mathbb{R}^m$ and $\text{Col } A$.

If both $\vec{b}_1, \vec{b}_2 \in \text{Col } A$, then $\exists\ \vec{u}_1$ & $\vec{u}_2$ s.t.

$$A\vec{u}_1 = \vec{b}_1$$
$$A\vec{u}_2 = \vec{b}_2$$

So

$$\vec{b}_1 + \vec{b}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2) \quad \text{so } \vec{b}_1 + \vec{b} \in \text{Col } A$$

Same for

$$\vec{b} = A\vec{u} \rightarrow c\vec{b} = A(c\vec{u}) \quad \text{so } c\vec{b} \in \text{Col } A$$

∎

When does $\text{Col } A = \mathbb{R}^m$?

If a spanning set of $\text{Col } A$ is also a spanning set of $\mathbb{R}^m$ and vica versa then $\text{Col } A = \mathbb{R}^m$.

- If A has m lin. indep. columns then $\text{Col } A = \mathbb{R}^m$
- If $A\vec{x} = \vec{b}$ is consistant $\forall\ \vec{b} \in \mathbb{R}^m$

Note that if $m = n$ then A will be invertible.

Transformations

A <u>linear transformation</u> T from a vector space V into a vector space W is a rule that assigns to each $\vec{x} \in V$ a unique vector $T(\vec{x}) \in W$ s.t.

$$1. \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \forall\ \vec{u},\vec{v} \in V$$
$$2. \quad T(c\vec{u}) = cT(\vec{u}) \qquad \forall\ \vec{u} \in V,\ c \in \mathbb{R}$$

If T represents a matrix transformation then it has a standard matrix A.

$$\vec{x} \to T(\vec{x}) \quad \text{is just} \quad \vec{x} \mapsto A\vec{x}$$

and the Range of T is just Col A. We call the space of vectors which T maps to $\vec{0}$ the <u>Kernel</u> of T (or null space of T)

	$T(\vec{x})$	$A\vec{x}$
null space	Ker T	Nul A
	Range T	Col A