4.3 Linearly independent sets; Bases

Just as for vectors in $\mathbb{R}^n$, we can define linear independence of vectors in a *general* vector space:

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is <u>linearly independent</u> if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution where $c_i = 0$, $i = 1, \dots, p$.

<u>Ex</u> If $\vec{v}_1 = \sin x$ and $\vec{v}_2 = \cos x$, is $\{\vec{v}_1, \vec{v}_2\}$ a linearly independent set?

Can we find c_1 and c_2 such that

$$c_1 \sin x + c_2 \cos x = \vec{0} \qquad \underline{\forall\, x?} \qquad \text{no.}$$

If sinx and cosx were
nearly dependent then one
them would be a scalar multiple
the other.

$\Rightarrow$ $\sin x$ and $\cos x$ form a linearly independent set

$\rightarrow$ It is important here that we consider all x — the dependence relation must hold for all choices of the parameter x.

<u>Ex</u> The set $\{\vec{0}\}$ is lin. dep. since $\exists$ many c such that

$$c\vec{0} = \vec{0}$$

<u>Thm</u> A set $\{\vec{v}_1, \dots, \vec{v}_p\}$ of two or more vectors, with $\vec{v}_1 \neq \vec{0}$, is linearly dependent iff some $\vec{v}_j$, $1 < j \leq p$, is a linear combination of $\vec{v}_1, \dots, \vec{v}_{j-1}$

Consider a vector space H spanned by the vectors $\vec{b}_1, \ldots, \vec{b}_q$:

- H is described by $\{\vec{b}_1, \ldots, \vec{b}_q\}$ in a spanning sense
- $\{\vec{b}_1, \ldots, \vec{b}_q\}$ may or may not be linearly independent.

We will find it useful to find a <u>minimum spanning set</u> for H; i.e. a set of lin. independent vectors which span H.

<u>Def</u> Let H be a subspace of a vector space V. A set of vectors $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ in V is a <u>basis</u> for H if

i) $\mathcal{B}$ is a linearly independent set

ii) $H = \text{span}\{\vec{b}_1, \cdots, \vec{b}_p\}$.

Q: does H have a unique basis?

<u>Ex</u> The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ form a <u>basis</u> for $\mathbb{R}^3$.
So do $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$.

A vector space may have more than one basis!

The vectors $\vec{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \vec{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$ form

the <u>standard basis of $\mathbb{R}^n$</u>.

Ex Is $\{1,\ t+1,\ t^2+1,\ t^2-t\}$ a basis for $\mathbb{P}_2$?

Since $t^2 - t = a \cdot 1 + b(t+1) + c(t^2+1)$
when $a=0,\ b=-1,\ c=1$

this set is linearly dependent, so it cannot be a basis. The set does, however, span $\mathbb{P}_2$.

→ Standard basis for $\mathbb{P}_n$ is $\{1, t, t^2, \dots, t^n\}$

idea If we have a set of vectors which span a space, we can obtain a basis from it by removing one-by-one vectors which are linear combinations of other vectors

Thm (Spanning Set)

Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a set in V and let $H = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$.

i) if one of the vectors $\vec{v}_k$ in S is a linear comb. of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

ii) if $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

Proof idea

a) We can rearrange the vectors so that $\vec{v}_p$ is dependent on the others: $\vec{v}_p = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{p-1}\vec{v}_{p-1}$. Then any $x \in \text{Span } S$ can be written $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_{p-1}\vec{v}_{p-1} + \alpha_p[c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{p-1}\vec{v}_{p-1}]$
so $\vec{x} = (\alpha_1 + c_1)\vec{v}_1 + (\alpha_2 + c_2)\vec{v}_2 + \dots + (\alpha_{p-1} + c_{p-1})\vec{v}_{p-1}$

b) Let $T = S$. If T has a vector dep. on others, remove it. Continue until only linearly independent vectors remain.

Now we seek bases for Nul A and Col A

Nul A is straight forward...

Write solution of $A\vec{x} = \vec{0}$ in parametric form and find a linear combination of vectors which span Nul A (Sections 1.5 and 4.2)

Col A

The column space is just the space spanned by the columns of A

$$A\vec{x} = \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\text{Span Col } A}$$

To find a basis for Col A, we begin with the columns of A and remove any that are linearly dependent on other columns.

EX $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Since this is in reduced echelon form, it is easy to see that columns 1, 2, 5 are lin. indep and columns 3, 4 are lin. dep on column 1 & 2.

thus a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Key observation 1: Elementary row operations do not change the linear dependence relations among the columns of a matrix.

Key observation 2: The pivot columns of a matrix A form a basis for Col A

Key observation 3: If A is factored into LU, the pivot columns of U are in the same locations as the pivot colums of A.

<u>Ex</u> Find a basis for

$$A = \begin{bmatrix} 2 & 3 & 1 & 8 & 1 \\ 4 & 6 & 3 & 18 & 5 \\ 0 & 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 14 & 5 \end{bmatrix}$$

A factors into

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 2 & 3 & 1 & 8 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & & \uparrow & \uparrow & \uparrow \\ 1 & & 3 & 4 & 5 \end{array} \end{array}$$

need 1, 3, 4, 5 colums <u>of A</u>, not the row equivalent U.

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 18 \\ 2 \\ 14 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 5 \end{bmatrix} \right\}$$ is a basis for Col A.

☆ Choose pivot columns of A, not pivot columns of U!