4.4 Coordinate Systems

<u>Thm</u> Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x}$ in V there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_n\vec{b}_n .$$

Proof. Suppose, in addition to the above, that $\vec{x} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$ (ie. another expansion exists). But then

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \dots + (c_n - d_n)\vec{b}_n$$

Which can only be true if $c_i - d_i = 0$, $i = 1, \dots, n$ since the $\vec{b}_i$ are linearly independent (why are they?) Thus $c_i = d_i$ so the expansion is unique.

These c_i are <u>the coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$.</u>

- Pick a basis $\Rightarrow \exists$ a unique set of coordinates specifies a vector using the chosen basis.

<u>Ex</u> The vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ can be expressed using the basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ as $3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus the coordinates are 3 and 4.

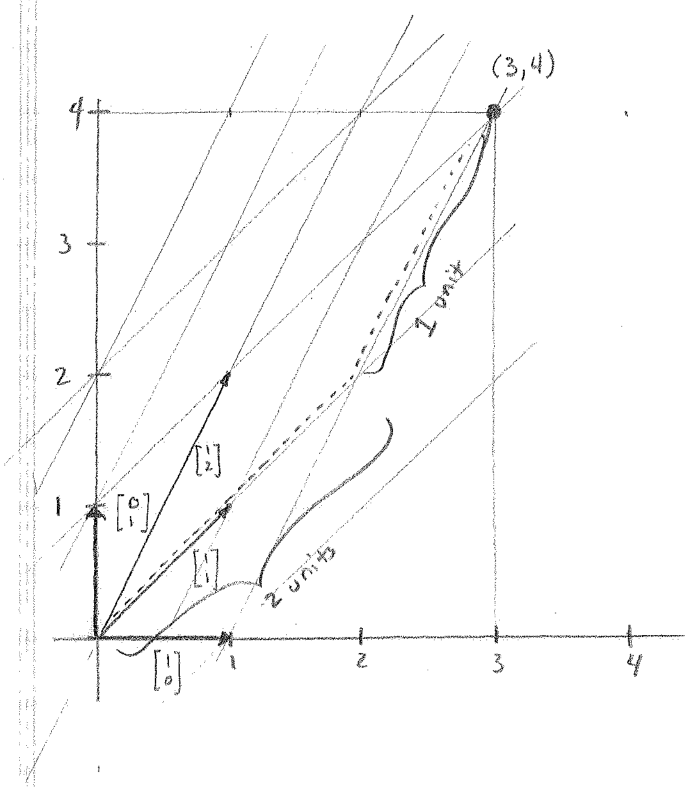
<u>Ex</u> Express $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ using the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

We find that $c_1 = 2$, $c_2 = 1$. So

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and 2,1 are the coordinates relative to the basis $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$



[Key: While any of a number of bases can be chosen, once one is chosen, every vector has a unique set of coordinates relative to that basis.]

Also – some bases are more "natural" than others.

Text uses notation $[\vec{x}]_{\mathcal{B}}$ to be the coordinate vector of $\vec{x}$ relative to the basis $\mathcal{B}$. So if $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ then

if $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(coord. vector w.r.t. standard basis for $\mathbb{R}^2$)

Notice that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

2nd basis vectors

relative to second basis

relative to standard basis

4.4

If we have $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ then the matrix

$P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$ is called the <u>change-of-coordinates matrix</u>.

$$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$$

$P_{\mathcal{B}}$ is invertible since it is square and has linearly independent columns, so

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}$$

$\boxed{\text{Coordinate mapping is one-to-one and onto.}}$

<u>Ex</u> $\mathcal{B} = \{1, t+1, t^2-t\}$ is a basis for $\mathbb{P}_2$.

Find the coordinate vector of $\vec{p}(t) = a + bt + ct^2$ relative to $\mathcal{B}$.

$$\begin{aligned} a + bt + ct^2 &= \alpha_1(1) + \alpha_2(t+1) + \alpha_3(t^2-t) \\ &= (\alpha_1+\alpha_2) + (\alpha_2-\alpha_3)t + \alpha_3 t^2 \end{aligned}$$

So $c = \alpha_3$

$b = \alpha_2 - \alpha_3 = \alpha_2 - c \rightarrow \alpha_2 = b + c$

$a = \alpha_1 + \alpha_2 = \alpha_1 + (b+c) \rightarrow \alpha_1 = a - (b+c) = a - b - c$

So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a-b-c \\ b+c \\ c \end{bmatrix}$

Ex Even simpler, $\mathcal{B} = \{1, t, t^2\}$. Then $a+bt+ct^2$

has $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

In both of these cases, we have gone from the vector space $\mathbb{P}_2$ to the space $\mathbb{R}^3$.

Thm Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be an ordered bases for a vector space V. Then the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a <u>one-to-one</u> linear transformation from V <u>onto</u> $\mathbb{R}^n$.

This means that the transformation $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is an <u>isomophism</u> from V to $\mathbb{R}^n$.

Two vector spaces that are related by an isomophism, while possibily very different in representation and appearence, require exactly the same amount of information to specify vectors within them, and operations performed on elements in one space have matching operations which can be performed on elements in the other space

Key here: <u>one-to-one</u> and <u>onto</u>.

Ex: Show $\mathbb{P}_3$ isomorphic to $\mathbb{R}^4$