4.5 Dimension of a Vector Space

Consider the vector space $\mathbb{R}^2$.

Is the set $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ lin. independent?

If $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a set of vectors from $\mathbb{R}^n$, how must p compare to n for all the $\vec{v}_i$ to be lin. indep?

$$\boxed{p \le n}$$

<u>Thm</u> If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Remember that V with the basis $\mathcal{B}$ as above is isomorphic to $\mathbb{R}^n$. Thus, while this is not a proof, it makes sense that the number of lin. indep. vectors which can be selected from each space should be the same.

<u>Thm</u> If a vector space V has a basis of n vectors, then every basis of V has exactly n vectors.

<u>Proof</u> assume $\mathcal{B}_1$ is a basis for V and has n vectors.
assume $\mathcal{B}_2$ is a basis for V but has m vectors

By the first thm today, $m \le n$ and $n \le m$
So the only possibility is $m = n$.

When considering vector spaces, keep in mind that they don't necessarily have to be spanned by a finite set.

Ex the space spanned by $\{\sin 0x, \sin 1x, \sin 2x, \dots, \sin jx, \dots\}$ needs an infinite number of vectors to span it.

Ex How many vectors are needed in a basis for $\mathbb{R}^n$? $\underline{n}$

Ex How many vectors are needed in a basis for $\mathbb{P}_n$? $\underline{n+1}$

We can ask this (useful) question about every vector space, which leads to the following definition:

Def The dimension of V is the number of vectors in a basis for V. The dimension of $\{\vec{0}\}$ is defined to be zero.

Ex The dimension of $\mathbb{R}^n$ is n: $n = \dim \mathbb{R}^n$

Ex The dimension of $\mathbb{P}_n$ is $n+1$ $n+1 = \dim \mathbb{P}_n$

Ex What is the dimension of the subspace H

$$H = \left\{ \begin{bmatrix} 2a+4b+c \\ a+2b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Any vector $\vec{v}$ in H can be expressed

$$\vec{v} = \begin{bmatrix} 2a+4b+c \\ a+2b \\ c \end{bmatrix} = \underbrace{a\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}}_{\text{lin. dep.}} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

or $\vec{v} = (a+2b)\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So H has a basis of 2 vectors $\rightarrow$ $\dim H = 2$

<u>Def</u> If V is spanned by a finite set then V is <u>finite-dimensional</u>, otherwise it is <u>infinite-dimensional</u>.

<u>Ex</u> $\mathbb{R}^n$ $\forall n$ are finite-dimensional vector spaces but the space spanned by

$$\{\cos nx : n = 0, 1, 2, \ldots\}$$

is infinite-dimensional. We will not usually be concerned with infinite-dimensional spaces in this course.

<u>Thm</u> Let H be a subspace of a finite-dimensional space V. Any set of linearly independent vectors in H can be expanded (augmented) using other vectors from H to form a basis for H. Also $\dim H \leq \dim V$.

Ex Suppose H is a subspace of V and both spaces are n-dimensional. Show $H = V$

Both H and V have bases with n vectors in them, and these n vectors are lin. indep.

The basis for H consists of n linearly independent vectors from V. Since $n = \dim V$, these n vectors must also form a basis for V. Similarly, the basis for V must also be a basis for H, so $H = V$.

Thm Let V be an n-dimensional vector space, $n \geq 1$, and take S to be a subset of V that contains exactly n elements:

1. if S is lin. indep. then S is a basis for V.
2. if S spans V then S is a basis for V.

Ex Find the dimension of Nul A and Col A if

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To identify Nul A we solve $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$ are vectors in Nul A

Since Nul A is spanned by 1 vector we see $\boxed{\dim \text{Nul} A = 1}$

To find dim Col A, we form $A\vec{x}$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = (x_1 - x_3)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Col A has a basis of 2 vectors, so dim Col A = 2

Given a matrix A, dim Nul A = # free varicbles in $A\vec{x} = \vec{0}$ and dim Col A = # pivot columns in A.