4.6 Rank

Given an $m \times n$ matrix A, we have discussed 2 associated spaces, Nul A and Col A. We can also describe Row A, the row space of A.

$$\text{Row } A = \text{Col } A^T$$

We will often write row vectors horizontally.

$$\vec{r} = (2, 1, 6, 8)$$

Which is equivalent to $\vec{r} = \begin{bmatrix} 2 \\ 1 \\ 6 \\ 8 \end{bmatrix}$. Remember, a $1 \times n$ <u>matrix</u> is

$$[a_1 \; a_2 \; \cdots \; a_n]$$

(no commas, square brackets)

<u>Thm</u> If two matrices A and B are row equivalent, they share the same row space. If B is in echelon form then the nonzero rows of B form a basis for this space.

<u>Ex</u> $$A = \begin{bmatrix} 1 & -2 & -4 & 3 \\ 2 & -3 & -5 & 6 \\ 4 & -7 & -15 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So a basis for Row A is $\{(1,0,2,0), (0,1,3,0), (0,0,0,1)\}$

Any echelon form of A can be used to find a basis for Row A and identify pivot columns of A to get a basis for Col A.

Proof of Thm. if $A \sim B$ then the rows in B are lin. comb. of the rows of A. Thus Span (rows of A) = Span (rows of B) so the spanned spaces must be equivalent.

Def The <u>Rank</u> of A is the dimension of the column space of A.

$$\text{Rank } A = \dim \text{Col } A$$

Thm If A is $m \times n$ then

1. $\dim \text{Col } A = \dim \text{Row } A$
2. $\text{rank } A + \dim \text{Nul } A = n$

Proof ① $\dim \text{Col } A = \#\text{ pivot columns} = \#\text{ pivot rows} = \dim \text{Row } A$.

② $\dim \text{Nul } A$ is the number of free variables in $A\vec{x} = \vec{0}$ which is the same as the number of nonpivot columns in A

$$\#\text{ pivot columns in } A + \#\text{ nonpivot columns} = \#\text{ columns}$$

Ex Suppose an 8×5 matrix has a 3 dimensional null space. What is the rank?

$$5 = 3 + \text{rank } A \rightarrow \boxed{\text{rank } A = 2}$$

Ex #19 $A\vec{x} = \vec{0}$ A is 5×6 Will $A\vec{x} = \vec{b}$ always be consistent?

$\dim \text{Nul } A = 1$

$\text{rank } A = 6 - 1 = 5$ yes!

Since A has 5 rows, Col A is a subspace of $\mathbb{R}^5$. Since $\text{rank } A = 5$ we conclude $\text{Col } A = \mathbb{R}^5$ $\rightarrow$ always consistent.

Now we can add some new items to the Invertible Matrix Theorem:

Thm A is $n \times n$. The following are either all true or are all false:

1) A is invertible
2) $A \sim I$
3) A is a product of elementary matrices
4) $A\vec{x} = \vec{0}$ has only trivial solution
5) $A\vec{x} = \vec{b}$ has a unique solution, $\forall \vec{b} \in \mathbb{R}^n$
6) $\exists$ $n \times n$ matrix C s.t. $AC = I$
7) $\exists$ $n \times n$ matrix D s.t. $DA = I$

$\Big\}$ $D = C = A^{-1}$

8) A^T is invertible

new
9) Columns of A form basis for $\mathbb{R}^n$
10) $\text{Col } A = \mathbb{R}^n$
11) $\dim \text{Col } A = n$
12) $\text{rank } A = n$
13) $\text{Nul } A = \{\vec{0}\}$
14) $\dim \text{Nul } A = 0$