4.7 Change of Basis

Given a vector space V and a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for V, we can describe any vector in V as a linear combination of the vectors in $\mathcal{B}$

$$\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n$$

where the vector $[\vec{v}]_{\mathcal{B}} = \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is the <u>coordinate vector</u> and $\vec{x} \in \mathbb{R}^n$

We know that for a particular $\vec{v} \in V$, there is a unique coordinate vector – for a particular basis...

Given another basis, say $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$, we have

$$\vec{v} = y_1\vec{c}_1 + y_2\vec{c}_2 + \cdots y_n\vec{c}_n$$

and $[\vec{v}]_{\mathcal{C}} = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is the coordinate vector of $\vec{v}$ relative to the basis $\mathcal{C}$, and, as before, $\vec{y} \in \mathbb{R}^n$.

Q1: what is the relationship between $\vec{x}$ and $\vec{y}$?

Q2: given $\vec{x}$, can we find $\vec{y}$ and vica versa?

A1: Both $\vec{x}$ and $\vec{y}$ are coordinates for $\vec{v}$ but wrt different bases.

A2: Well, let's see...

Ex Consider $\mathbb{R}^2$. If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

then

$$\vec{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So $[\vec{v}]_{\mathcal{B}} = \vec{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

Now, what is necessary to be able to find $[\vec{v}]_{\mathcal{C}}$?

$$\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [\vec{v}]_{\mathcal{B}} \qquad \vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} [\vec{v}]_{\mathcal{C}}$$

So

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [\vec{v}]_{\mathcal{B}}$$

So

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [\vec{v}]_{\mathcal{B}}$$

$$[\vec{v}]_{\mathcal{C}} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} [\vec{v}]_{\mathcal{B}}$$

$$[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} [\vec{v}]_{\mathcal{B}}$$

So $[\vec{v}]_{\mathcal{C}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Check: $\vec{v} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ ✓

In *general*: Both $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \cdots, \vec{c}_n\}$ are bases for $\mathbb{R}^n$. Then if

$$\vec{v} = [\vec{b}_1 \cdots \vec{b}_n][\vec{v}]_{\mathcal{B}} = [\vec{c}_1 \cdots \vec{c}_n][\vec{v}]_{\mathcal{C}}$$

We find

$$[\vec{v}]_{\mathcal{C}} = [\vec{c}_1 \cdots \vec{c}_n]^{-1}[\vec{b}_1 \cdots \vec{b}_n][\vec{v}]_{\mathcal{B}}$$

or

$$[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{v}]_{\mathcal{B}}$$

where

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [\vec{c}_1 \cdots \vec{c}_n]^{-1}[\vec{b}_1 \cdots \vec{b}_n].$$

Notice that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = [\vec{b}_1 \cdots \vec{b}_n]^{-1}[\vec{c}_1 \cdots \vec{c}_n]$ and

$$[\vec{v}]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\vec{v}]_{\mathcal{C}}$$

So $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$

To compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ we can reduce

$$[\vec{c}_1 \cdots \vec{c}_n \,\vdots\, \vec{b}_1 \cdots \vec{b}_n] \sim [I \,\vdots\, \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}]$$

(think of row op. as mult. by elementary matrices)

For a general vector space V, assume 2 bases

$$\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\} \qquad \mathcal{C} = \{\vec{c}_1, \cdots, \vec{c}_n\}$$

for any $\vec{v} \in V$

$$\vec{v} = x_1 \vec{b}_1 + \cdots + x_n \vec{b}$$

$$\begin{aligned}[\vec{v}]_{\mathcal{C}} &= [x_1 \vec{b}_1 + \cdots + x_n \vec{b}_n]_{\mathcal{C}} \\ &= x_1 [\vec{b}_1]_{\mathcal{C}} + \cdots + x_n [\vec{b}_n]_{\mathcal{C}} \\ &= \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right] [\vec{v}]_{\mathcal{B}}\end{aligned}$$

If $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \cdots [\vec{b}_n]_{\mathcal{C}} \right]$ then

$$[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\vec{v}]_{\mathcal{B}}$$

Key idea: trying to find coordinate vectors for $\vec{v}$ w.r.t. $\mathcal{C}$, need vectors in $\mathcal{B}$ expressed in terms of vectors in $\mathcal{C}$, hence $[\vec{b}_i]_{\mathcal{C}}$ are the columns in $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

Example: Let $\mathcal{B} = \{1+t^2,\ t-2t^2,\ 1-t\}$

$\mathcal{C} = \{1,\ t,\ t^2\}$

A) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

$\vec{b}_1 = 1\cdot 1 + 0\cdot t + 1\cdot t^2 \rightarrow [\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{b}_2 = 0\cdot 1 + 1\cdot t - 2\cdot t^2 \rightarrow [b_2]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\vec{b}_3 = 1\cdot 1 - 1\cdot t + 0\cdot t^2 \rightarrow [\vec{b}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$P_{\mathcal{C}\leftarrow B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

B) Find $[5-2t]_{\mathcal{B}}$

Note that $[5-2t]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$

$\therefore\ [5-2t]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$

$\therefore\ P_{\mathcal{C}\leftarrow\mathcal{B}}\, [5-2t]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ Solve to find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

So $[5-2t]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Check $2(1+t^2) + 1(t-2t^2) + 3(1-t)$.

$= 2 + 2t^2 + t - 2t^2 + 3 - 3t$

$= 5 - 2t + 0t^2$ ✓

4.7

#13 $\mathbb{P}_2$ $\mathcal{B} = \{1-2t+t^2,\ 3-5t+4t^2,\ 2t+3t^2\}$

$\mathcal{C} = \{1,\ t,\ t^2\}$

Find $\underset{C\leftarrow B}{P}$

Need $\underset{C\leftarrow B}{P} = \left[\ [\vec{b}_1]_C\ \ [\vec{b}_2]_C\ \ [\vec{b}_3]_C\ \right]$

$[\vec{b}_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ since $(1)1 + (-2)t + (1)t^2 = \vec{b}_1$

$[\vec{b}_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

$[\vec{b}_3]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\rightarrow$ $\boxed{\underset{C\leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}}$

Consider $\vec{v} = -1+2t$. $[\vec{v}]_C = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} [\vec{v}]_B \rightarrow [\vec{v}_B] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \boxed{[\vec{v}]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

Check $5(1-2t+t^2) - 2(3-5t+4t^2) + 1(2t+3t^2)$

$= 5-6 \quad -10t + 10t + 2t \quad +5t^2 - 8t^2 + 3t^2$

$= -1 \quad +2t \quad = -1+2t$ ✓