4.8 Application to Difference Equations

We usually measure discrete data wrt time
- record temp. every minute
- etc.

Thus, a continuous function is only recorded at finite intervals. These intervals may be small enough to "appear" continuous

CD player 44,100 times a second
samples only magnitude - freq. information is recorded implicitly and is related to the sampling rate.

Note that the sound qual. depends on 2 things
- sampling rate
- resolution of samples

The <u>signal</u> is the function whose output can be mapped to the integers to produce a (possibly infinite) list of values. Text denotes the vector space containing signals as $\mathbb{S}$.

Suppose we have 2 signals from $\mathbb{S}$, $\{y_k\}$, $\{w_k\}$ where the k subscript denotes an integer index. Thus one signal is

$$y_0, y_1, y_2, \dots, y_n, \dots$$

The 2 signals will be lin. indep if $c_1 y_k + c_2 w_k = 0$
$\forall\ y_k$ and w_k

Thus, if they are lin. indep. then

$$c_1 y_k + c_2 w_k = 0$$
$$c_1 y_{k+1} + c_2 w_{k+1} = 0$$

are both true only if $c_1 = c_2 = 0 \quad \forall\ k$

$$\underbrace{\begin{bmatrix} y_k & w_k \\ y_{k+1} & w_{k+1} \end{bmatrix}}_{\text{Casorati matrix.}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If the casorati matrix is invertible then this system has only the trivial solution. Note that we only need to find a particular k if that yields an invertible matrix. To show the signals are lin. dep. we would have to show the system had a nontrivial soln $\forall\ k$.

A linear difference eq of order n has the form

$$a_0 y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \qquad \forall\ k$$

Where $a_0, \ldots, a_n$ are constants and $a_0 \neq 0$, $a_n \neq 0$. Here $\{z_k\}$ is the signal which is described as a linear combination of inputs (the y_k).

If $z_k = 0 \ \forall k$ then the eq. is homogeneous, otherwise it is non-homogeneous.

Ex Solve $y_{k+2} - 5y_{k+1} - 6y_k = 0$, i.e. find a function y_k which makes this true for all k.

We will assume $y_k = r^k$ for some r. Then

$$y_{k+1} = r^{k+1} \quad \text{and} \quad y_{k+2} = r^{k+2}.$$

So

$$r^{k+2} - 5r^{k+1} - 6r^k = r^k(r^2 - 5r - 6) = 0$$

From this we see that either $r^k = 0 \to r = 0$ (trivial soln)

or $r^2 - 5r - 6 = 0$

$\uparrow$ auxiliary eq. (characteristic eq)

$$r^2 - 5r - 6 = (r+1)(r-6) = 0 \quad \to \quad r = -1, \ r = 6$$

So both $(-1)^k$ and 6^k are solutions. Are they lin. independent?

$$\begin{bmatrix} (-1)^k & 6^k \\ (-1)^{k+1} & 6^{k+1} \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ -1 & 6 \end{bmatrix} \text{ if } k=0$$

$$\det \begin{vmatrix} 1 & 1 \\ -1 & 6 \end{vmatrix} = 6 + 1 = 7 \neq 0$$ invertible so $(-1)^k$ and 6^k are lin. independent.

Also, not to hard to show that any lin. comb. of these functions is also a solution

$$y_k = c_1 (-1)^k + c_2 6^k$$

This says that $(-1)^k$ and 6^k form a basis for the solution space.

If the roots are repeated then things need to be adjusted a bit - we won't see any here. If the roots are complex conjugates $r = s \pm i\omega$ then the solutions are of the form $s^k \cos \omega k$ and $s^k \sin \omega k$.

Thm 16

If $a_n \neq 0$ and $\{z_k\}$ is given then

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k \quad \forall k$$

has a unique solution whenever $y_0, \ldots, y_{n-1}$ are specified.

Thm 17

The set H of all solutions of the n^{th} order homogeneous lin. diff. eq.

$$y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = 0 \quad \forall k$$

is an n-dimensional vector space

Where is the Linear Algebra?

- determine if Casorati matrix is invertible
- reduction of n^{th} order lin. diff. eq to 1^{st} order system

<u>Ex</u> $y_{k+2} + 3y_{k+1} - 10y_k = 0$ 2^{nd} order $\Rightarrow$ system of 1^{st} order.

so

$$y_{k+2} = 10y_k - 3y_{k+1}$$

So $\begin{bmatrix} y_{k+1} \\ y_{k+2} \end{bmatrix} = \begin{bmatrix} y_{k+1} \\ 10y_k - 3y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & -3 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$

if $\vec{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$ then $\vec{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \end{bmatrix}$ so

$\vec{x}_{k+1} = A\vec{x}_k$ where $A = \begin{bmatrix} 0 & 1 \\ 10 & -3 \end{bmatrix}$

$\vec{x}_1 = A\vec{x}_0$

$\vec{x}_2 = A\vec{x}_1 = AA\vec{x}_0 = A^2\vec{x}_0$

$\vdots$

$\vec{x}_k = A^k\vec{x}_0$

See pg 284-285 (3rd ed)