4.9 Applications to Markov Chains

Suppose a college has 3000 students. From one year to the next 4% of off-campus students move on-campus, and 5% of on-campus students move off-campus. Initially 40% of the students live off-campus and 60% are on-campus. Develop a linear equation that calculates these percentages for the following year.

Let n = # students on-campus, f = # students off-campus

$$\begin{aligned}\text{\# students off-campus next year} &= f - 0.04f + 0.05n \\ &= (1-0.04)f + 0.05n \\ &= 0.96f + 0.05n\end{aligned}$$

$$\begin{aligned}\text{\# students on-campus next year} &= n - 0.05n + 0.04f \\ &= (1-0.05)n + 0.04f \\ &= 0.04f + 0.95n\end{aligned}$$

$$\underbrace{\begin{bmatrix} f \\ n \end{bmatrix}}_{\text{next year}} = \begin{bmatrix} 0.96f + 0.05n \\ 0.04f + 0.95n \end{bmatrix} = \begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix} \underbrace{\begin{bmatrix} f \\ n \end{bmatrix}}_{\text{this year}}$$

Here $P = \begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix}$ is the migration matrix

To find next years percentages we need $\begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix} \underbrace{\begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}}_{\text{this years percentages}}$

The matrix P has a special property — the sum of the elements in each column is 1.

A square matrix with non-negative entries whose columns sum to 1 is called a stochastic matrix

A vector with non-negative elements that sum to 1 is called a probability vector.

In our example P is a stochastic matrix and

$$V_0 = \begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}$$ is a probability vector.

It makes sense, given our college with a fixed enrollment, that if $\vec{V}_0$ is the pop. dist. for the initial year then $\vec{V}_1 = M\vec{V}_0$ will also be a prob. vector since the pop. is just shifting around somewhat

In general, if P is a stochastic matrix and $\vec{V}_k$ are probability vectors, then

$$\vec{V}_{k+1} = P\vec{V}_k \qquad k = 0, 1, 2, \ldots$$

describes the generation of vectors in a <u>Markov Chain</u>

<u>Exer #3</u> healthy students → 95% healthy tomorrow
ill students → 55% ill tomorrow

a) Stochastic matrix

$$P = \begin{array}{c} \\ \left[\begin{array}{cc} 0.95 & 0.45 \\ 0.05 & 0.55 \end{array}\right] \\ \underbrace{\text{well} \qquad \text{ill}}_{\text{\% of current}} \end{array} \begin{array}{l} \left.\begin{array}{l} \text{well} \\ \text{ill} \end{array}\right\} \text{tomorrow} \\ \\ \end{array}$$

b) $\overrightarrow{\text{Monday}} = \begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix}$

$\overrightarrow{\text{Tuesday}} = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}\begin{bmatrix} 0.80 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 0.85 \\ 0.15 \end{bmatrix}$ 15% ill

$\overrightarrow{\text{Wednesday}} = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}\begin{bmatrix} 0.85 \\ 0.15 \end{bmatrix} = \begin{bmatrix} 0.875 \\ 0.125 \end{bmatrix}$ 12.5% ill

c) $\overrightarrow{Today} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\overrightarrow{Tomorrow} = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$$

$$\overrightarrow{2days} = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix} \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.925 \\ 0.075 \end{bmatrix}$$

Prob. of well 2 days from now: 0.925

Q: Suppose that the stochastic matrix for our College problem remained the same for many years. What happens to the pop. of the schools as the years go by?

$$\begin{array}{c|cccccc} k & 0 & 5 & 10 & 20 & 30 & 50 \\ \hline \vec{v}_k = & \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} & \begin{bmatrix} 0.46 \\ 0.54 \end{bmatrix} & \begin{bmatrix} 0.49 \\ 0.51 \end{bmatrix} & \begin{bmatrix} 0.53 \\ 0.47 \end{bmatrix} & \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} & \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} \end{array}$$

Looks like it might be approaching a limit.

If $\vec{q}$ is the limiting value, then $P\vec{q}$ should just yield $\vec{q}$. So

$$P\vec{q} = \vec{q}$$

but then $P\vec{q} - \vec{q} = (P-I)\vec{q} = \vec{0}$

If we find a solution to this homogeneous system then we have found our limiting value...

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So $$\left(\begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0.96-1 & 0.05 & 0 \\ 0.04 & 0.95-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.04 & 0.05 & 0 \\ 0.04 & -0.05 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} -0.04 & 0.05 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad -0.04\, q_1 + 0.05\, q_2 = 0$$

$$\text{or} \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{0.05}{0.04} q_2 \\ q_2 \end{bmatrix} = q_2 \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}$$

Thus $\begin{bmatrix} 5/4 \\ 1 \end{bmatrix}$ forms a basis for the Nul (P-I). if $q_2 = 4$ then

$$\vec{q} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

or, to get a prob. vector

$$\vec{q} = \begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix} \simeq \begin{bmatrix} 0.555556 \\ 0.444444 \end{bmatrix}$$

In the limit, # off-campus students is $3000 \cdot 5/9 \simeq 1{,}667$

\# on-campus students is $3000 \cdot 4/9 \simeq 1{,}333$

Thm 18 guarentees that this approach works for any starting probability vector $\vec{V}_0$