5.1 Eigenvalues and Eigenvectors

Consider the iteration

$$\vec{x}_{k+1} = A\vec{x}_k$$

which we saw in both application sections in the last chapter.

$$\vec{x}_1 = A\vec{x}_0 \quad , \quad \vec{x}_2 = A\vec{x}_1 = A^2\vec{x}_0 \quad , \quad \vec{x}_3 = A\vec{x}_2 = A^3\vec{x}_0 \, , \; \ldots$$

so

$$\vec{x}_k = A^k\vec{x}_0$$

Computing $\vec{x}_{10000}$ or even $\vec{x}_{100}$ can be very time consuming.

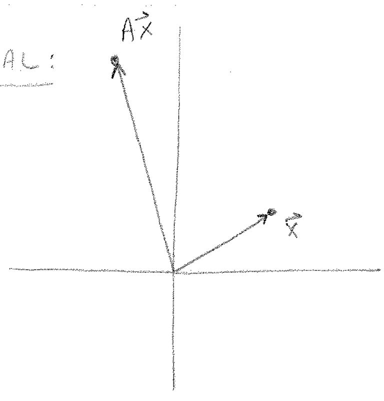
Suppose that we knew a scalar λ and vector $\vec{x}_0$ such that $A\vec{x}_0 = \lambda\vec{x}_0$. Then

$$\vec{x}_k = \lambda^k \vec{x}_0$$

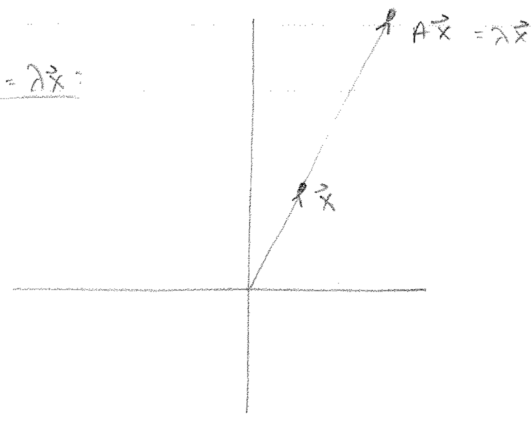
Which is easier to work with. In fact if $|\lambda| < 1$ then $\vec{x}_k \to \vec{0}$ as $k \to \infty$ and if $|\lambda| > 1$ then the entries in $\vec{x}_k$ grow without bound.

What must happen for $A\vec{x} = \lambda\vec{x}$? Since $\lambda\vec{x}$ is just a scaling of $\vec{x}$ the vector $A\vec{x}$ must be parallel to $\vec{x}$.

IN GENERAL:



IF $A\vec{x} = \lambda\vec{x}$:



Suppose A is an $n \times n$ matrix and $\exists$ a nonzero vector $\vec{x}$ and a scalar λ such that $A\vec{x} = \lambda\vec{x}$. Then $\vec{x}$ is an

* <u>eigenvector</u> of A and λ is the associated <u>eigenvalue</u>.

Ex: show that $\lambda = -1$ is an eigenvalue of $\begin{bmatrix} 4 & 2 \\ 10 & 3 \end{bmatrix}$ and find the associated eigen vector.

$$\begin{bmatrix} 4 & 2 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-1)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4+1 & 2 \\ 10 & 3+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving: $\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 10 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2/5\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2/5 \\ 1 \end{bmatrix}$$ if $x_2 = 5$ then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

The fact that we could find an eigenvector shows that $\lambda = -1$ is an eigenvalue. This was apparent as soon as we realized that

$$\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$$

had a non-empty null space.

* eigenwert: German for "proper value"

Given A, how do we find the values of λ associated with it? This can be hard...

$$A\vec{x} = \lambda\vec{x} \rightarrow A\vec{x} - \lambda\vec{x} = 0 \rightarrow (A - \lambda I)\vec{x} = 0$$

So we need to find values of λ such that the matrix $(A - \lambda I)$ has a non-empty null space.

In the next section we will see how to compute the eigenvalues of a matrix.

The null space of $(A - \lambda I)$ is called the eigenspace of A corresponding to λ.

Ex Find the eigenspace of $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$, $\lambda = 3$.

$$A\vec{x} = 3\vec{x} \rightarrow (A - 3I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 4-3 & 2 & 3 & 0 \\ -1 & 1-3 & -3 & 0 \\ 2 & 4 & 9-3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 + 2x_2 + 3x_3 = 0 \rightarrow x_1 = -2x_2 - 3x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace.

Suppose A has $\lambda = 0$ as an eigenvalue. Then

$$A\vec{x} = 0\vec{x} \to A\vec{x} = \vec{0}$$

So the corresponding eigenvector is a solution of the homogeneous eq. Since $\vec{0}$ can never be an eigen vector (why not?) we conclude that $A\vec{x} = \vec{0}$ must have a nontrivial solution.

Thus A is not invertible if $\lambda = 0$ is an eigenvalue.

If A is invertible then $A\vec{x} = \vec{0}$ has only the trivial solution so that $\lambda = 0$ is not an eigenvalue.

Thm The $n \times n$ matrix A is invertible iff zero is not an eigenvalue of A.

(This can be added to the list of statements in the Invertible Matrix Thm.)

Ex Find the eigenvalues of $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

Which will have a non triv. solution iff at least one column is not a pivot column.

Thus if $\lambda = 3$, $\lambda = 1$, or $\lambda = 6$ $(A - \lambda I)$ will be noninvertible and an eigenvector can be found for each λ.

Thm: If $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A then $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent.

Proof: Assume $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly dependent. Then $\exists\ \vec{v}_{p+1}$ s.t. $\vec{v}_{p+1} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p \quad (*)$, $p<r$

(i.e. assume $\{\vec{v}_1, \dots, \vec{v}_r\}$ is lin. dep. but $\{\vec{v}_1, \dots, \vec{v}_p\}$ is lin. independent)

$$A\vec{v}_{p+1} = A(c_1\vec{v}_1 + \dots + c_p\vec{v}_p)$$
$$\lambda_{p+1}\vec{v}_{p+1} = c_1\lambda_1\vec{v}_1 + \dots c_p\lambda_p\vec{v}_p \quad (**)$$

$$(**) - \lambda_{p+1}(*) \rightarrow \quad \vec{0} = c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \dots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p$$

$\lambda_i - \lambda_{p+1} \neq 0$, $i = 1, \dots, p$ since λ are <u>distinct</u>

$\{\vec{v}_1, \dots, \vec{v}_p\}$ are lin. indep. So we see that

$$c_i = 0, \quad i = 1, \dots, p$$

Then $\vec{v}_{p+1} = \vec{0}$ which is not possible so proof by contradiction is complete.

Difference Equations: $\quad \vec{x}_{k+1} = A\vec{x}_k$

If $\vec{x}_0 = \vec{v}$ is an eigenvector of A associated with λ then

$$\vec{x}_1 = A\vec{x}_0 = \lambda\vec{v}$$
$$\vec{x}_2 = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$$
$$\vdots$$
$$\vec{x}_k = \lambda^k\vec{v}$$

Try with $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$; λ_i associated with $\vec{v}_i$