5.2 The Characteristic Equation

<u>Ex</u> Find eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

We seek λ and vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$

or

$$(A - \lambda I)\vec{x} = 0$$

So

$$A - \lambda I$$

must be noninvertible $\rightarrow$ $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4 = 0$$

$$10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6 = 0 \qquad (*)$$

$$(\lambda - 1)(\lambda - 6) = 0$$

So $\lambda = 1$, $\lambda = 6$

Here, $(*)$ is called the <u>Characteristic Eq.</u> It is

$$\det(A - \lambda I) = 0$$

for any square $n \times n$ matrix A. In theory this eq. can be used to find eigenvalues: but it can be difficult to solve when $n > 2$.

Ex Find the char. eq. of $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\det \begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)\left[(-1-\lambda)(4-\lambda) - 0\right] = 0$$
$$= (2-\lambda)(-1-\lambda)(4-\lambda) = 0$$
$$= -(\lambda-2)(\lambda+1)(\lambda-4) = 0$$

So $\lambda = 2$, $\lambda = -1$, $\lambda = 4$ are the eigenvalues.

Ex Find the eigenvalues of I_5

$$\det(I_5 - \lambda I_5) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^5 = 0$$

So $\lambda = 1$ is a root 5 times - it is an eigenvalue of I_5 with multiplicity 5.

Ex $\begin{bmatrix} 3 & 2 & 1 & 2 & 1 \\ 0 & 1 & -1 & 8 & 5 \\ 0 & 0 & 3 & 6 & -1 \\ 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ has eigenvalues 3, 1, 3, 2, 3

or 3, 3, 3, 1, 2

or 3 (mult. 3), 1, 2

Notice that an $n \times n$ matrix will have n eigenvalues, counting multiplicities, because of the Fundamental Theorem of Algebra

Similarity

Consider $B = P^{-1}AP$. What are the eigenvalues of B?

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda I \\ &= P^{-1}AP - \lambda P^{-1}P \\ &= P^{-1}(A - \lambda I)P \end{aligned}$$

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}) \det(A - \lambda I) \det(P) \\ &= \det(P^{-1}P) \det(A - \lambda I) \\ &= \det(A - \lambda I) \end{aligned}$$

So the same values of λ satisfy both

$$\det(B - \lambda I) = 0$$
$$\det(A - \lambda I) = 0$$

Thus the eigenvalues of A and B are the same.

If $\exists$ an invertible matrix P such that

$$B = P^{-1}AP \qquad (**)$$

for $n \times n$ matrices A and B, then A is <u>similar</u> to B and $(**)$ is a <u>similarity transform</u>.

Are the eigenvectors the same?

$B\vec{x} = \lambda\vec{x} \longrightarrow$ eigenvectors are $\vec{x}$

$P^{-1}AP\vec{x} = \lambda\vec{x}$

$AP\vec{x} = \lambda P\vec{x} \longrightarrow$ eigenvectors are $P\vec{x}$

different! (but related)

Similarity is theoretically and practically useful. If a matrix A is similar to a diagonal (or triangular) matrix whose eigenvalues are easy to determine, then the eigenvalues of A are easy to determine.

☆ Note that being similar and being row equivalent are two very different things.

Ex Show that $\begin{bmatrix} -61 & 40 \\ -96 & 63 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$$\det \begin{bmatrix} -61-\lambda & 40 \\ -96 & 63-\lambda \end{bmatrix} = (-61-\lambda)(63-\lambda) + 40 \cdot 96 = 0$$

$$= -3843 - 2\lambda + \lambda^2 + 3840 = 0$$

so $\quad \lambda^2 - 2\lambda - 3 = 0$

$$(\lambda-3)(\lambda+1) = 0 \qquad \lambda = 3, \ \lambda = -1$$

Same eigenvalues as $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ so they are similar

(in this case $P = \begin{bmatrix} 3 & -2 \\ -8 & 5 \end{bmatrix}$)

How do we find P? (stay tuned...)

Remember our college population example?

Recall $\vec{X}_{k+1} = A\vec{X}_k$ with $A = \begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix}$ $\vec{X}_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

Find the eigenvalues of A

$$\det(A - \lambda I) = \det \begin{bmatrix} 0.96-\lambda & 0.05 \\ 0.04 & 0.95-\lambda \end{bmatrix} = 0$$

$$(0.96-\lambda)(0.95-\lambda) - (0.04)(0.05) = 0$$
$$\lambda^2 - 1.91\lambda + 0.912 - 0.002 = 0$$
$$\lambda^2 - 1.91\lambda + 0.91 = 0$$

$$\lambda = \frac{1.91 \pm \sqrt{3.6481 - 3.64}}{2} = \frac{1.91 \pm 0.09}{2}$$

$$\lambda = \frac{1.82}{2} = 0.91 \quad \text{or} \quad \lambda = \frac{2}{2} = 1$$

Find eigenvectors

$\lambda = 0.91$ $\quad A - \lambda I = \begin{bmatrix} 0.05 & 0.05 \\ 0.04 & 0.04 \end{bmatrix}$

$$\begin{bmatrix} 0.05 & 0.05 \\ 0.04 & 0.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 = -x_2 \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 1$ $\quad A - \lambda I = \begin{bmatrix} -0.04 & 0.05 \\ 0.04 & -0.05 \end{bmatrix}$ $\qquad 0.04x_1 = 0.05x_2$

$x_1 = 1.25x_2$ $\qquad \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

So we have $\lambda_1 = 0.91$ $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1$ $\vec{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, they form a basis for $\mathbb{R}^2$ and we can express our initial condition as a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$c_1 = -1.4/9 = -7/45$

$c_2 = 1/9 = 5/45 = 1/9$

Now $\vec{x}_1 = A\vec{x}_0 = A\left[c_1\vec{v}_1 + c_2\vec{v}_2\right] =$
$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2$
$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$

so $\vec{x}_n = A^n\vec{x}_0 = c_1 A^n \vec{v}_1 + c_2 A^n \vec{v}_2$
$= c_1 \lambda_1^n \vec{v}_1 + c_2 \lambda_2^n v_2$

For our problem

$$\vec{x}_n = \left(-\frac{7}{45}\right)(0.91)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left(\frac{5}{45}\right)(1)^k \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

as $k \to \infty$ $(0.91)^k \to 0$ so $\vec{x}_n \to \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.\overline{5} \\ 0.\overline{4} \end{bmatrix}$

Long term - behavior