5.3 Diagonalization

An $n \times n$ matrix A is <u>diagonalizable</u> if it is similar to a diagonal matrix

If $\exists$ an $n \times n$ matrix P s.t. $A = PDP^{-1}$ where D is diagonal, then A is diagonizable.

Recall that if $A = PDP^{-1}$ then A and D have the same eigenvalues, and since D is diagonal, its eigenvalues are just its diagonal elements.

Notice:
$$A^2 = (PDP^{-1})^2 = PDP^{-1}PDP^{-1}$$
$$= PD(P^{-1}P)DP^{-1}$$
$$= PDIDP^{-1}$$
$$= PD^2P^{-1}$$

$$A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$$

$$\boxed{A^k = PD^kP^{-1}}$$

If $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$

$D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}$

Thm

An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.

Proof: Suppose A is diagonizable, then $A = PDP^{-1}$ so that $AP = PD$. Then

$$A[\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n] = [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$[A\vec{p}_1 \ A p_2 \cdots A\vec{p}_n] = [\lambda_1\vec{p}_1 \ \lambda_2\vec{p}_2 \cdots \lambda_n\vec{p}_n]$$

So

$$A\vec{p}_i = \lambda_i \vec{p}_i$$

Since the columns of P are linearly independent, we see that A has n linearly independent eigenvectors.

Now suppose A has n linearly independent eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$ so that

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

Then

$$[A\vec{x} \ A\vec{x}_2 \cdots A\vec{x}_n] = [\lambda_1\vec{x}_1 \ \lambda_2\vec{x}_2 \cdots \lambda_n\vec{x}_n]$$

$$A[\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n] = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n]\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda \end{bmatrix} = [\vec{x}_1 \ \vec{x}_2 \cdots \vec{x}_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$AX = XD$$

Since X has n linearly independent columns it is invertible so

$$A = XDX^{-1}$$ ✓

5.3

Let's try and diagonalize a matrix, i.e. find P such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \qquad n=2$$

1. Find λ:

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(4-\lambda) - 3 = 0$$

$$8 - 6\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0 \quad \rightarrow \quad \lambda = 1, \quad \lambda = 5$$

2. Find n linearly independent eigenvectors:

$$(A-\lambda I)\vec{x} = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2-\lambda & 1 & 0 \\ 3 & 4-\lambda & 0 \end{array}\right]$$

$\lambda = 1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 5$

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

3. Construct P

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$$

4. Construct D

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

eigenvalues must appear in the same order here as the eigenvectors in P.

5. Check

$$PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{-4} \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{4} \begin{bmatrix} -1 & 5 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{4} \begin{bmatrix} -8 & -4 \\ -12 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = A \checkmark$$

Note: can avoid computing P^{-1} by checking $AP = PD$.

If n linearly independent eigenvectors cannot be found in step 2, then the matrix is not diagonalizable.

Thm If an $n \times n$ matrix A has n distinct eigenvalues then A is diagonalizable.

n distinct $\lambda \Rightarrow$ n linearly independent eigenvectors

Note that a matrix may not have n distinct eigenvalues but still be diagonizable

(This happens when repeated eigenvalues have eigenspaces with dimension greater than one)

Thm Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a) For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k

b) A is diagonizable iff $\sum \dim(\text{eigenspace } \lambda_k) = n$. This is possible only if $\dim(\text{eigenspace } \lambda_k) = \text{mult}(\lambda_k)$.

c) If A is diagonizable and B_k is a basis for the eigenspace of λ_k then $\bigcup_{i=1}^{p} B_k$ forms an eigenvector basis for $\mathbb{R}^n$.