5.4 Eigenvectors and Linear Transformations

Consider the linear transformation $T: V \to W$ from an n-dimensional vectorspace V to an m-dimensional vector space W.

Recall that if $\mathcal{B}$ is an ordered basis for V then a vector $\vec{x} \in V$ can be represented by $[\vec{x}]_{\mathcal{B}}$, it's coordinate vector relative to $\mathcal{B}$, and $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^n$.

If $\mathcal{C}$ is an ordered basis for W then $[\vec{y}]_{\mathcal{C}}$ is a coordinate vector relative to $\mathcal{C}$. $[\vec{x}]_{\mathcal{C}} \in \mathbb{R}^m$

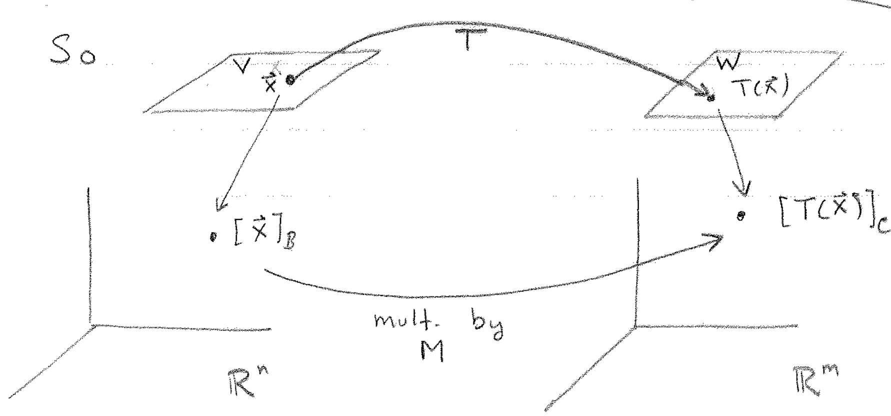
Suppose $\vec{x} \in V$ and $\vec{x} = r_1\vec{b}_1 + \cdots + r_n\vec{b}_n$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

then $\underbrace{T(\vec{x})}_{\text{in } W} = r_1 T(\vec{b}_1) + \cdots + r_n T(\vec{b}_n)$

$$[T(\vec{x})]_{\mathcal{C}} = r_1 [T(\vec{b}_1)]_{\mathcal{C}} + \cdots + r_n [T(\vec{b}_n)]_{\mathcal{C}}$$

$$[T(\vec{x})]_{\mathcal{C}} = \underbrace{\begin{bmatrix} [T(\vec{b}_1)]_{\mathcal{C}} & \cdots & [T(\vec{b}_n)]_{\mathcal{C}} \end{bmatrix}}_{M} [\vec{x}]_{\mathcal{B}}$$

So



matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$

Ex #4. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ basis for V. $T: V \to \mathbb{R}^2$

$$T(x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (standard basis for $\mathbb{R}^2$)

$T(\vec{b}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $T(\vec{b}_2) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$, $T(\vec{b}_3) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

So $[T(\vec{b}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $[T(\vec{b}_2)]_{\mathcal{C}} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$, $[T(\vec{b}_2)]_{\mathcal{C}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$$\therefore \quad M = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

Thus $[T(\vec{x})]_C = M[\vec{x}]_B$

In many cases $W = V$ so $T: V \to V$. In this case we expect the matrix M to be square since both the domain and codomain have the same dimension.

$T: V \to V$. In this case we can use $\mathcal{B}$ as a basis for $T(\vec{x})$ as well as for $\vec{x}$.

In this case the matrix M is $n \times n$ and is called the <u>$\mathcal{B}$-matrix</u> for T, denoted $[T]_{\mathcal{B}}$. Thus

$$[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}} [x]_{\mathcal{B}}$$

$$\begin{tikzcd}
\vec{x} \arrow[r, "T"] \arrow[d] & T(\vec{x}) \arrow[d] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "\text{mult. by}", "{[T]_{\mathcal{B}}}"'] & {[T(\vec{x})]_{\mathcal{B}}}
\end{tikzcd}$$

<u>Ex</u> (Ex 2 in text)

$T: \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$

(first derivative)

If $\mathcal{B} = \{1, t, t^2\}$ then we can find $[T]_{\mathcal{B}}$ ($\mathcal{B}$-matrix):

$T(1) = 0$ $\quad [1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad [T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(t) = 1$ $\quad [t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(t^2) = 2t$ $\quad [t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

↑ vectors in $\mathcal{B}$ ↖ images of vectors in $\mathcal{B}$

So

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, to compute the derivative of $\vec{p} = 3 - 2t + 4t^2$ we find

$$[\vec{p}]_B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \text{and} \quad \text{compute}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

Sure enough, $[\vec{p}\,']_\rho = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$ ✓

Now suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is computed with $\vec{x} \mapsto A\vec{x}$.

If A is diagonalizable then it has n lin. indep. eigenvectors. These eigenvectors form a basis for $\mathbb{R}^n$.

Let $\mathcal{B}$ = { eigenvectors of a diagonalizable matrix A }
$= \{ \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n \}$

$$\vec{x} = r_1 \vec{p}_1 + r_2 \vec{p}_2 + \cdots + r_n \vec{p}_n \qquad [x]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$\begin{aligned} T(\vec{x}) &= r_1 T(\vec{p}_1) + \cdots r_n T(\vec{p}_n) \\ &= r_1 A\vec{p}_1 + \cdots + r_n A\vec{p}_n \\ &= r_1 \lambda_1 \vec{p}_1 + \cdots + r_n \lambda_n \vec{p}_n \end{aligned} \qquad [T(\vec{x})]_B = \begin{bmatrix} r_1\lambda_1 \\ \vdots \\ r_n\lambda_n \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = D\,[\vec{x}]_B \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

So D is the $\mathcal{B}$-matrix.

Now, since $AP = A[\vec{p}_1 \cdots \vec{p}_n]$

$$= [A\vec{p}_1 \cdots A\vec{p}_n]$$

$$= [\lambda_1\vec{p}_1 \cdots \lambda_n\vec{p}_n]$$

$$= [\vec{p}_1 \cdots \vec{p}_n] D$$

$$= PD$$

So $A = PDP^{-1}$

What does this all mean? The $\{\vec{p}_i\}$ is in some sense, a more convienent basis in $\mathbb{R}^n$ than the standard basis $\{\vec{e}_i\}$ when dealing with the transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by $\vec{x} \mapsto A\vec{x}$

$\{\vec{e}_i\}$	$\{\vec{p}_i\}$
$\vec{x} \mapsto A\vec{x}$	$\vec{u} \mapsto D\vec{u}$
$n \times n$ matrix generally full	$n \times n$ diagonal matrix
$\Rightarrow n^2$ multiplications	$\Rightarrow n$ multiplications