5.5 Complex Eigenvalues

Consider $R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ which is a rotation and scaling matrix. Find λ such that $R\vec{x} = \lambda\vec{x}$.

$$\det(R - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(1-\lambda) + 1 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2} = 1 \pm \frac{\sqrt{-4}}{2} = 1 \pm \frac{2i}{2} = 1 \pm i$$

$$\lambda = 1 \pm i$$

Eigen values of real matrices will always occur in conjugate pairs (fund. Thm of Algebra)

Now, find eigenvectors:

$\lambda = 1 + i$: $\quad (R - \lambda I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-ix_1 - x_2 = 0 \quad \text{and} \quad x_1 - ix_2 = 0$$

Choose $x_1 = ix_2$ so $\quad x_2 = 1 \rightarrow x_1 = i$

$$\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$ (any scalar multiple works as well)

We can find $\vec{x}$ corresponding to $\lambda = 1-i$ in the same way, or note the following when A is a real matrix

$$A\vec{x} = \lambda\vec{x}$$
$$\overline{A\vec{x}} = \overline{\lambda\vec{x}}$$
$$\overline{A}\,\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$$
$$A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$$

So if λ is a complex eigenvalue of A with $\vec{x}$ its associated eigenvector, then $\overline{\lambda}$ is also an eigenvalue and $\overline{\vec{x}}$ is the corresponding eigenvector

so $\lambda = 1+i$ $\quad \vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = 1-i$ $\quad \overline{\vec{x}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

We will use $\operatorname{Re}\vec{x}$ to denote the real part of $\vec{x}$ if $\vec{x}$ has complex entries. Similarly $\operatorname{Im}\vec{x}$ is the imaginary part

$$\vec{x} = \operatorname{Re}\vec{x} + i\operatorname{Im}\vec{x}$$

$$\vec{x} = \begin{bmatrix} 3+i \\ 4 \end{bmatrix} \qquad \operatorname{Re}\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \operatorname{Im}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find eigenvalues of $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$\det \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} = 0 \Rightarrow (a-\lambda)^2 + b^2 = 0$$

$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0$$

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$$
$$= a \pm \frac{\sqrt{-4b^2}}{2}$$
$$= a \pm \frac{2bi}{2}$$
$$= a \pm bi$$

Interesting!

Thm Let A be a 2×2 real matrix with a complex eigenvalue $\lambda = a - ib$ and associated eigenvector $\vec{v} \in \mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad P = [\operatorname{Re}\vec{v} \;\; \operatorname{Im}\vec{v}], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This shows that multiplication by a 2×2 matrix with complex eigenvalue always produces a rotation.

Since $A = PCP^{-1}$, $A\vec{x} = P[C[P^{-1}\vec{x}]]$

change variables

rotate and scale

change back to original variables

Proof: First we note two things

① $A\vec{v} = A(\mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v})$
$= A\,\mathrm{Re}\,\vec{v} + i\,A\,\mathrm{Im}\,\vec{v}$

so

$\mathrm{Re}\,A\vec{v} = A\,\mathrm{Re}\,\vec{v}$ and $\mathrm{Im}\,A\vec{v} = A\,\mathrm{Im}\,\vec{v}$

② $(a-ib)\vec{v} = (a-ib)(\mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v}) = \underbrace{a\,\mathrm{Re}\,\vec{v} + b\,\mathrm{Im}\,\vec{v}}_{\mathrm{Re}} + i\,\underbrace{(a\,\mathrm{Im}\,\vec{v} - b\,\mathrm{Re}\,\vec{v})}_{\mathrm{Im}}$

Now the proof: Since we want to show $A = PCP^{-1}$ it is enough to show $AP = PC$ and that P is invertible.

$$
\begin{aligned}
AP &= A[\mathrm{Re}\,\vec{v} \quad \mathrm{Im}\,\vec{v}] \\
&= [A\,\mathrm{Re}\,\vec{v} \quad A\,\mathrm{Im}\,\vec{v}] \\
&= [\mathrm{Re}\,A\vec{v} \quad \mathrm{Im}\,A\vec{v}] && \text{by ①} \\
&= [\mathrm{Re}\,(a-ib)\vec{v} \quad \mathrm{Im}\,(a-ib)\vec{v}] && \text{since } a-ib \text{ is an eigenvalue of } A \\
&= [a\,\mathrm{Re}\,\vec{v} + b\,\mathrm{Im}\,\vec{v} \quad a\,\mathrm{Im}\,\vec{v} - b\,\mathrm{Re}\,\vec{v}] && \text{by ②} \\
&= [\mathrm{Re}\,\vec{v} \quad \mathrm{Im}\,\vec{v}] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\
&= PC
\end{aligned}
$$

How do we know P is invertible?

Ex Find P & C such that $A = PCP^{-1}$ if $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

$$0 = \det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) + 2$$

$$= 15 - 8\lambda + \lambda^2 + 2$$

$$0 = \lambda^2 - 8\lambda + 17$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4 \cdot 17}}{2} = \frac{8 \pm i2}{2} = 4 \pm i$$

So $\lambda = a - ib = 4 - i$.

$$(A - \lambda I)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 5-(4-i) & -2 \\ 1 & 3-(4-i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1+i)x_1 - 2x_2 = 0$$
$$x_1 + (-1+i)x_2 = 0$$

if $x_2 = 1$ then $(1+i)x_1 = 2 \rightarrow x_1 = 1 - i$

So

$$\vec{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \quad \mathrm{Re}\,\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathrm{Im}\,\vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}}$$

Check $AP = PC$

$$AP = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$$

same ✓

$$PC = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & -1 \end{bmatrix}$$