5.6 Dynamical Systems

Consider $A = \begin{bmatrix} .9 & 0 \\ 0 & .7 \end{bmatrix}$ We know the eigenvalues are

$\lambda_1 = 0.9 \qquad \lambda_2 = 0.7$

We can also find the eigenvectors - they are

$$\vec{v}_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If this matrix were part of an iteration $\vec{x}_{k+1} = A\vec{x}_k$ we can use the eigenvalues and eigenvectors to study its behavior.

To carry out the iteration we must begin with an initial vector $\vec{x}_0$. Express this in terms of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$$ (ask why this is always possible)

Now, compute $\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots$

$$\vec{x}_1 = A\vec{x}_0 = A\,[c_1\vec{v}_1 + c_2\vec{v}_2] = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$$
$$\vec{x}_2 = A\vec{x}_1 = A\,[c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2] = c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\vec{x}_k = A\vec{x}_{k-1} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda^k\vec{v}_2$$

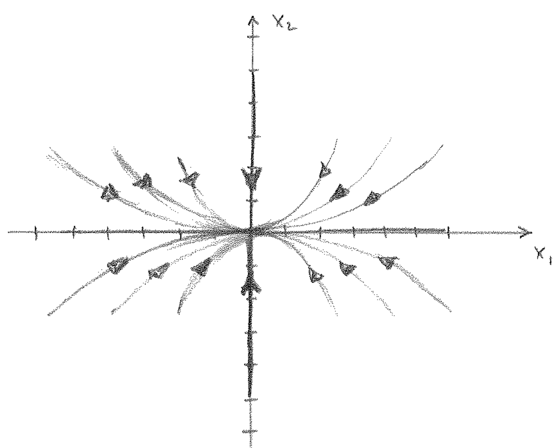
Thus, the value of any $\vec{x}_k$ can be computed easily by raising λ_1 and λ_2 to the k^{th} power and doing a linear combination.

For our example, we find $\vec{x}_k = c_1(.9)^k\begin{bmatrix}1\\0\end{bmatrix} + c_2(.7)^k\begin{bmatrix}0\\1\end{bmatrix}$

Q: What does $\vec{x}_k$ approach as $k\to\infty$?

since $(.9)^k\to 0$ and $(.7)^k\to 0$ we expect $\vec{x}_k \to \vec{0}$. This happens regardless of any $\vec{x}_0 = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ vector.

In fact, since $(.7)^k \to 0$ faster than $(.9)^k\to 0$, we expect that the second coordinate of $\vec{x}_k$ will approach zero more rapidly than the first as $k\to\infty$.



k	$(0.9)^k$	$(0.7)^k$
0	1	1
1	.9	.7
2	.81	.49
3	.73	.34
4	.66	.24
5	.59	.17
6	.53	.12
7	.48	.08
8	.43	.06
9	.39	.04

In this case, the origin is an <u>attractor</u>. No matter where one starts, as $k\to\infty$ you end up at the origin

Notice that we always approach the origin (in this example) along the x_1 axis ($\vec{v}_1$ eigenvector) <u>unless</u> we start on the x_2 axis (along $\vec{v}_2$).

This is because $\lambda_2 = 0.7 < \lambda_1 = 0.9$.

Now consider $A = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda_1 = 1.5$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

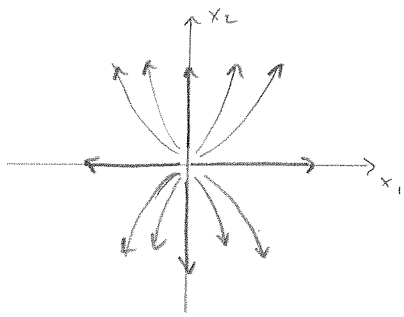
$\vec{x}_{k+1} = A\vec{x}_k$ with $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ gives us again

$$\vec{x}_k = c_1\lambda_1^k\,\vec{v}_1 + c_2\lambda_2^k\,\vec{v}_2$$

$$\vec{x}_k = c_1(1.5)^k\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(2)^k\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1.5)^k \\ c_2\,2^k \end{bmatrix}$$

Now the origin is a <u>repellor</u> since for any $\vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we are pushed away from the origin as k increases.

Since (in this example) the x_2 coordinate increases the fastest, we increasingly move in that direction as $k \to \infty$



If $|\lambda_i| < 1$ then we will be attracted toward the origin in the $\vec{v}_i$ direction

If $|\lambda_i| > 1$ then we are repelled from the origin in the $\vec{v}_i$ direction

Suppose we mixed these two ...

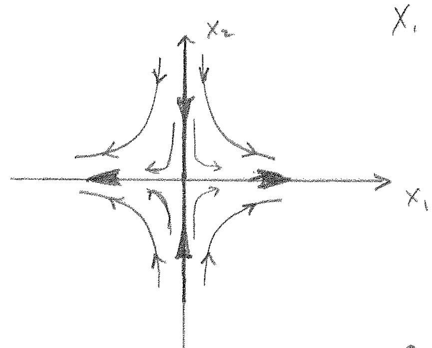
$$A = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix} \qquad \lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0.5 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We expect to move toward the origin along $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ but move away from the origin along $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Here again $\vec{x}_k = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$

$$\vec{x}_1 = c_1 (2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 2^k \\ c_1 (0.5)^k \end{bmatrix}$$



Now the origin is a <u>saddle point</u>.

If $c_1 = 0$ then we start on the x_2 axis and we will be attracted to the origin. However, if we start even a little off the x_2 axis we end up moving away from the origin with the x_1 axis as an asymptote.

In fact, for large k $\quad \vec{x}_k \simeq 2^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

All of this work has been with diagonal matrices. If the matrix is not diagonal then the outcome will be similar except that the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ will not (in general) point along the axes.

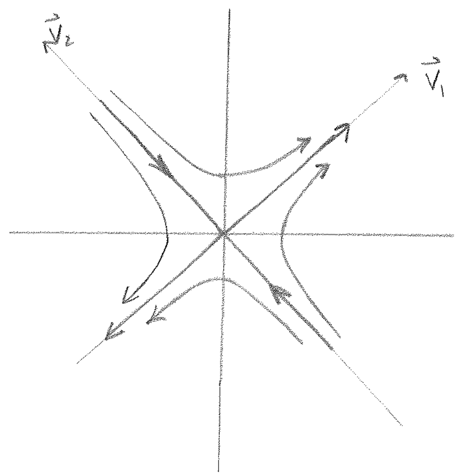
5.6

$$A = \begin{bmatrix} 1.25 & .75 \\ .75 & 1.25 \end{bmatrix} \qquad \lambda_1 = 2 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = .5 \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{cases} \vec{x}_{k+1} = A\vec{x}_k \\ \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{cases} \text{ gives } \quad \vec{x}_k = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

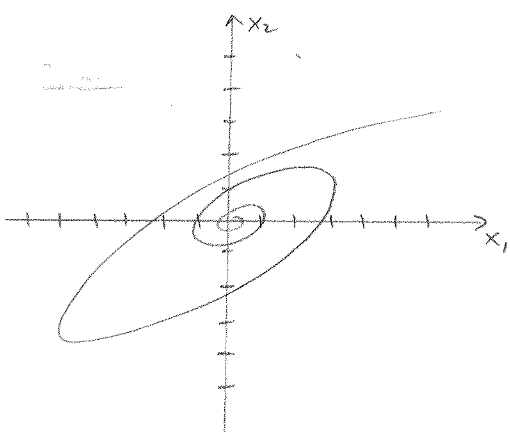
$$\vec{x}_k = c_1 (2)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 (.5)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Same behavior as last time, just different directions.

Finally, consider $A = \begin{bmatrix} 2.3 & -2.5 \\ 1 & -0.7 \end{bmatrix}$ $\quad \lambda_1 = 0.8 + 0.5i \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0.6 - 0.2i \end{bmatrix}$

$$\lambda_2 = 0.8 - 0.5i \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0.6 + 0.2i \end{bmatrix}$$



Complex eigenvalues and eigenvectors lead to solutions that spiral about the origin

If $Re\,\lambda < 0 \Rightarrow$ spiral in to origin

If $Re\,\lambda > 0 \Rightarrow$ spiral out away from origin

If $Re\,\lambda = 0 \Rightarrow$ rotate around origin in an ellipse.