6.1 Inner Product, Length & Orthogonality

Consider $\mathbb{R}^n$ and regard $\vec{x}$ as an $n \times 1$ matrix so that $\vec{x}^T$ is a $1 \times n$ matrix

1. $\vec{x}^T\vec{y}$ is 1×1 (a scalar)
2. $\vec{x}\vec{y}^T$ is $n \times n$ ($n \times n$ matrix)

The <u>inner product</u> (dot or scalar product) of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Properties $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, c is a scalar

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$
4. $\vec{u} \cdot \vec{u} \geq 0$ $\quad (\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0})$

<u>Ex</u> $\quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$

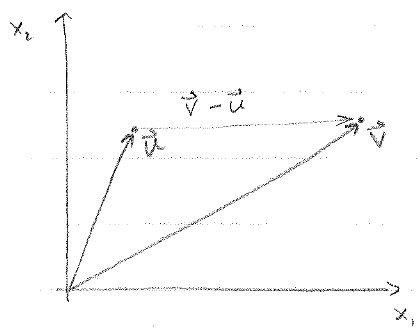
$$\vec{u} \cdot (2\vec{v}) = 2\vec{u} \cdot \vec{v} = 2\left[(3)(2) + (1)(0) + (-3)(6)\right]$$
$$= 2[6 - 18] = \boxed{-24}$$

$$\vec{u} \cdot \vec{u} = 3^2 + 1^2 + (-3)^2 = 9 + 1 + 9 = \boxed{19}$$

The <u>length</u> (magnitude or norm) of $\vec{v} \in \mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\vec{v}\cdot\vec{v}}$$

In $\mathbb{R}^2$ or $\mathbb{R}^3$ this computes the Euclidian distance between the origin and $\vec{v}$.



Distance between $\vec{u}$ & $\vec{v}$:

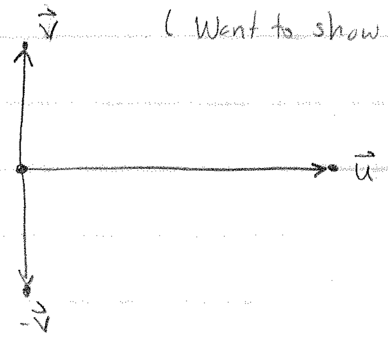
$$\|\vec{v}-\vec{u}\| = \sqrt{(v_1-u_1)^2 + (v_2-u_2)^2}$$

<u>Ex</u> If $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ what is $\|c\vec{v}\|$

$$\begin{aligned} \|c\vec{v}\| &= \sqrt{(c\vec{v})\cdot(c\vec{v})} \\ &= \sqrt{c^2\, \vec{v}\cdot\vec{v}} \\ &= |c|\sqrt{\vec{v}\cdot\vec{v}} \\ &= |c|\,\|\vec{v}\| \end{aligned}$$

$$\boxed{\|c\vec{v}\| = |c|\,\|\vec{v}\|}$$

6.1



(Want to show $\vec{u}\cdot\vec{v}=0$ iff $\vec{u}$ and $\vec{v}$ are orthogonal)

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$
$$= (\vec{u}-\vec{v})\cdot\vec{u} - (\vec{u}-\vec{v})\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} - \vec{v}\cdot u - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

may omit if time is tight

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= \|\vec{u}\|^2 + \|V\|^2 + 2\vec{u}\cdot\vec{v}$$

If $\vec{u}$ and $\vec{v}$ are Orthogonal (Perpendicular) then

$$\|\vec{u}-\vec{v}\|^2 = \|u+v\|^2$$
$$\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} = \|u\|^2 + \|v\|^2 + 2\vec{u}\cdot\vec{v}$$
$$-2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v} \quad \Rightarrow \quad \vec{u}\cdot\vec{v} = 0$$

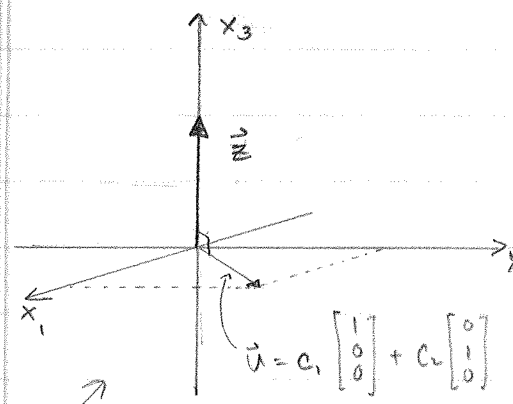
<u>Def</u> If $\vec{u},\vec{v} \in \mathbb{R}^n$, then $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = 0$

<u>Ex</u> Show that every $\vec{u} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix}$ is <u>orthogonal</u> to

$$\vec{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{u}\cdot\vec{z} = \begin{bmatrix}c_1\\c_2\\0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_1 & c_2 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = c_1\cdot 0 + c_2\cdot 0 + 0\cdot 1$$
$$= 0 \checkmark$$

In this last example, every vector in the x_1x_2-plane is orthogal to $\vec{z}$ (in the x_3 direction).



If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for a space W Then every vector in W is orthogonal to $\vec{z}$.

W is x_1x_2-plane

We say $\vec{z}$ is orthogonal to W

The set of all vectors which are orthogonal to W is denoted $W^{\perp}$, the <u>orthogonal complement</u> of W.

1. A vector $\vec{x}$ is in $W^{\perp}$ iff it is orthogonal to every vector in W

2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$

<u>Thm</u> Let A be $m \times n$. Then

$$(\text{Row } A)^{\perp} = \text{Nul } A$$
$$(\text{Col } A)^{\perp} = \text{Nul } A^T$$

<u>Proof</u>: if $A\vec{x} = \vec{0}$ then $\vec{x}$ is orthogonal to every row of A. So $\vec{x} \in \text{Nul} A$ iff $\vec{x} \in (\text{Row} A)^{\perp}$.
Repeat with $A^T\vec{y} = \vec{0}$ for second part.

6.1

Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Find $W^{\perp}$

need $\begin{bmatrix} 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$

and $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$

So solve $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$v_1 = 0$
$v_2 + v_4 = 0$

$v_1 = 0$
$v_2 = -v_4$
$v_3 = v_3$
$v_4 = v_4$

2 free variables
Pick $v_3 = c_1$
$v_4 = c_2$

$$\vec{v} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So $W^{\perp} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$