6.2 Orthogonal Sets

The set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an <u>orthogonal set</u> if each vector in the set is orthogonal to all other vectors in the set, i.e. $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$.

<u>Ex</u> $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthogonal set in $\mathbb{R}^3$

$$\left.\begin{aligned}
\vec{e}_1 \cdot \vec{e}_2 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \\
\vec{e}_1 \cdot \vec{e}_3 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \\
\vec{e}_2 \cdot \vec{e}_3 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0
\end{aligned}\right\}$$

<u>Thm</u> If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of non zero vectors in $\mathbb{R}^n$, then S is linearly independent and is a basis for the subspace spanned by S.

Proof $\vec{0} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$

$$\vec{0} \cdot \vec{u}_i = (c_1\vec{u}_1 + \cdots + c_p\vec{u}_p) \cdot \vec{u}_i$$
$$= c_1\vec{u}_1 \cdot \vec{u}_i + \cdots + c_i\vec{u}_i \cdot \vec{u}_i + \cdots + c_p\vec{u}_p \cdot \vec{u}_i$$

Since S is an orthogonal set, $\vec{u}_j \cdot \vec{u}_i = 0$ if $i \neq j$ so

$$\vec{0} \cdot \vec{u}_i = 0 = c_i\, \vec{u}_i \cdot \vec{u}_i$$

Since $\vec{u}_i \cdot \vec{u}_i \neq 0$ we find $c_i = 0$ for all $i = 1, 2, \ldots, p$.
Therefore, S is a linearly independent set and forms a basis for span $\{S\}$.

Def.
An orthogonal set S in $\mathbb{R}^n$ is an orthogonal basis for the subspace $W = \text{span}\{S\}$. If S has n vectors then $W = \mathbb{R}^n$

Thm Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then $\forall\ \vec{y} \in W$

$$\vec{y} = c_1\vec{u}_1 + \dots + c_p\vec{u}_p$$

and

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} \qquad j = 1, \dots, p$$

Proof

$$\vec{y}\cdot\vec{u}_j = (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{u}_j$$
$$= c_j\,\vec{u}_j\cdot\vec{u}_j$$

So $$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$$ ✓

Ex
Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}$. S is an orthogonal set.
Since S has 3 elements, all from $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

Express $\vec{y} = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$ as a linear combination of the elements of S.

$$\begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{u}_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 6 \qquad \vec{y} \cdot \vec{u}_1 = \begin{bmatrix} 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 20$$

$$\vec{u}_2 \cdot \vec{u}_2 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 5 \qquad \vec{y} \cdot \vec{u}_2 = \begin{bmatrix} 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 9$$

$$\vec{u}_3 \cdot \vec{u}_3 = \begin{bmatrix} -5 & -2 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 30 \qquad \vec{y} \cdot \vec{u}_3 = \begin{bmatrix} 3 & -5 & 7 \end{bmatrix} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 2$$

$$c_1 = \frac{20}{6} = \frac{10}{3} \qquad c_2 = \frac{9}{5} \qquad c_3 = \frac{2}{30} = \frac{1}{15}$$

$$\vec{y} = \frac{10}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{9}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

Note that we did this without "solving" a linear system.

If, in addition to being orthogonal, all the vector in S are normalized ($\|\vec{u}_i\| = 1$) Then the set is <u>orthonormal</u>. In this case S forms an <u>orthonormal basis</u> for $W = \text{Span}\{S\}$.

A vector can be normalized by scaling it by the reciprocal of its magnitude.

<u>Ex</u> $T = \left\{ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -5/\sqrt{30} \\ -2/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix} \right\}$ is an orthonormal set

formed by dividing $\vec{u}_i$ by $\sqrt{\vec{u}_i \cdot \vec{u}_i}$

"Fun" fact: If P is an $m \times n$ matrix with orthogonal columns, then $P^T P$ is a diagonal $n \times n$ matrix.

$$P = [\vec{p}_1 \;\; \vec{p}_2 \;\cdots\; \vec{p}_n] \qquad P^T = \begin{bmatrix} \vec{p}_1^{\,T} \\ \vec{p}_2^{\,T} \\ \vdots \\ \vec{p}_n^{\,T} \end{bmatrix}$$

$$P^TP = \begin{bmatrix} \vec{p}_1^{\,T} \\ \vec{p}_2^{\,T} \\ \vdots \\ \vec{p}_n^{\,T} \end{bmatrix} [\vec{p}_1 \;\; \vec{p}_2 \;\cdots\; \vec{p}_n]$$

$$= \begin{bmatrix} \vec{p}_1^{\,T}\vec{p}_1 & \vec{p}_1^{\,T}\vec{p}_2 & \cdots & \vec{p}_1^{\,T}\vec{p}_n \\ \vec{p}_2^{\,T}\vec{p}_1 & \vec{p}_2^{\,T}\vec{p}_2 & & \vdots \\ \vdots & & \ddots & \\ \vec{p}_n^{\,T}\vec{p}_1 & \vec{p}_n^{\,T}\vec{p}_2 & \cdots & \vec{p}_n^{\,T}\vec{p}_n \end{bmatrix}$$

Since the $\vec{p}_i$ are orthogonal $p_i^T p_j = 0$ if $i \neq j$

So

$$P^TP = \begin{bmatrix} \vec{p}_1^{\,T}\vec{p}_1 & & & \\ & \vec{p}_2^{\,T}\vec{p}_2 & & \\ & & \ddots & \\ & & & \vec{p}_n^{\,T}\vec{p}_n \end{bmatrix}$$

Now suppose U is an $m \times n$ matrix with orthonormal columns.
Then

$$\vec{u}_i^{\,T} \cdot \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

So $\quad U^TU = I \quad !$

Thm Let U be $m \times n$ and have orthonormal columns.
If $\vec{x}, \vec{y} \in \mathbb{R}^n$ then

1. $\|U\vec{x}\| = \|\vec{x}\|$
2. $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$
3. $(U\vec{x}) \cdot (U\vec{y}) = 0$ iff $\vec{x} \cdot \vec{y} = 0$

Proof $(U\vec{x})\cdot(U\vec{y}) = (U\vec{x})^T(U\vec{y})$

$$= \vec{x}^T U^T U \vec{y}$$
$$= \vec{x}^T \vec{y}$$
$$= \vec{x} \cdot \vec{y}$$

The Mapping $\vec{x} \mapsto U\vec{x}$ preserves length and orthogonality

Def If U is square $(n \times n)$ and has orthonormal columns then U is called an orthogonal matrix.

$$U^T U = I \quad \text{and} \quad U U^T = I$$

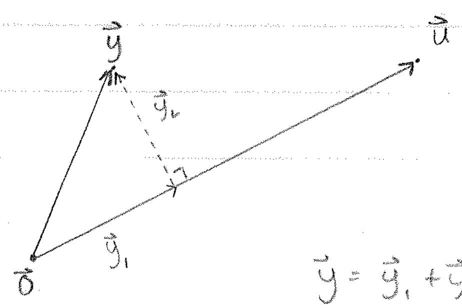
So we know what the inverse matrix of U is: U^T

$$U^T = U^{-1}.$$

Ex Solve $U\vec{x} = \vec{y}$ if U is an orthogonal matrix

$$\vec{x} = U^{-1}\vec{y} = U^T\vec{y}.$$

Orthogonal Projections



$$\vec{y} = \vec{y}_1 + \vec{y}_2$$

Suppose we want to "decompose" $\vec{y}$ into the sum of 2 vectors, one in the direction of $\vec{u}$ and one orthogonal to $\vec{u}$.

$$\vec{y} = \alpha \vec{u} + \vec{y}_2$$

$\nwarrow$ orth. to $\vec{u}$

$$\vec{y} \cdot \vec{u} = (\alpha \vec{u} + \vec{y}_2) \cdot \vec{u}$$
$$= \alpha \vec{u} \cdot \vec{u} + \vec{y}_2 \cdot \vec{u}.$$
$$= \alpha \vec{u} \cdot \vec{u}$$

So $\alpha = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

$\vec{y}_1$ = Component of $\vec{y}$ in direction of $\vec{u}$ $= \alpha \vec{u} = \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

$\vec{y}_2 = \vec{y} - \alpha \vec{u} = \vec{y} - \dfrac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

<u>Ex</u> $\quad \vec{y} = \begin{bmatrix} 4 \\ 10 \\ 7 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Write $\vec{y}$ as the sum of a vector $\parallel$ to $\vec{u}$ and one orthogonal to $\vec{u}$

$$\vec{y}_1 = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{4 \cdot 3 + 10 \cdot 1 + 7 \cdot 2}{3 \cdot 3 + 1 \cdot 1 + 2 \cdot 2} \vec{u} = \frac{12 + 10 + 14}{9 + 1 + 4} \vec{u} = \frac{36}{14} \vec{u} = \frac{18}{7} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{y}_2 = \begin{bmatrix} 4 \\ 10 \\ 7 \end{bmatrix} - \frac{18}{7} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 28 \\ 70 \\ 49 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 54 \\ 18 \\ 36 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -26 \\ 52 \\ 13 \end{bmatrix} = \frac{13}{7} \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$\therefore \quad \vec{y} = \frac{18}{7} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \frac{13}{7} \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$