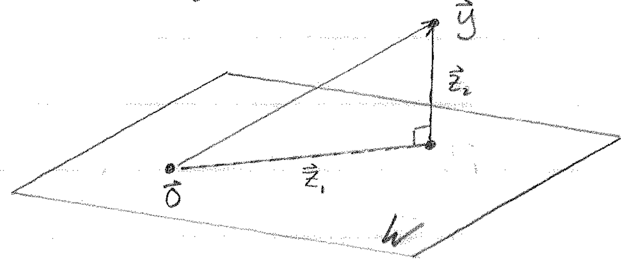
6.3 Orthogonal Projections



$\vec{y} = \vec{z}_1 + \vec{z}_2$

$\vec{z}_1 \in W \qquad \vec{z}_2 \in W^\perp$

<u>Thm</u> Let W be a subspace of $\mathbb{R}^n$ that has an orthogonal basis. Then each $\vec{y} \in \mathbb{R}^n$ can be written <u>uniquely</u> in the form

$$\vec{y} = \vec{z}_1 + \vec{z}_2$$

where $\vec{z}_1 \in W$ and $\vec{z}_2 \in W^\perp$.

If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is any orthogonal basis of W then

$$\vec{z}_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

$$\vec{z}_2 = \vec{y} - \vec{z}_1$$

$\vec{z}_1$ is the orthogonal projection of $\vec{y}$ onto W.

$$\boxed{\vec{z}_1 = \text{Proj}_W \vec{y}}$$

<u>Ex</u> $W = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} \right\} \qquad \vec{y} = \begin{bmatrix} 23 \\ -20 \\ 11 \end{bmatrix}$

Write $\vec{y}$ as the sum of 2 vectors, one in W and the other in $W^\perp$.

$\vec{y} = \vec{z}_1 + \vec{z}_2 \qquad$ Know $z_1 = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$

$$\vec{y}\cdot\vec{u}_1 = \begin{bmatrix}23 & -20 & 11\end{bmatrix}\begin{bmatrix}2\\0\\-3\end{bmatrix} = 13 \qquad \vec{u}_1\cdot\vec{u}_1 = 13$$

$$\vec{y}\cdot\vec{u}_2 = \begin{bmatrix}23 & -20 & 11\end{bmatrix}\begin{bmatrix}6\\3\\4\end{bmatrix} = 122 \qquad \vec{u}_2\cdot\vec{u}_2 = 61$$

So $$\vec{z}_1 = \frac{13}{13}\begin{bmatrix}2\\0\\-3\end{bmatrix} + \frac{122}{61}\begin{bmatrix}6\\3\\4\end{bmatrix} = \begin{bmatrix}2\\0\\-3\end{bmatrix} + 2\begin{bmatrix}6\\3\\4\end{bmatrix} = \begin{bmatrix}14\\6\\5\end{bmatrix}$$

$$\vec{z}_2 = \begin{bmatrix}23\\-20\\11\end{bmatrix} - \begin{bmatrix}14\\6\\5\end{bmatrix} = \begin{bmatrix}9\\-26\\6\end{bmatrix}$$

So $$\boxed{\vec{y} = \underbrace{\begin{bmatrix}14\\6\\5\end{bmatrix}}_{\text{in } W} + \underbrace{\begin{bmatrix}9\\-26\\6\end{bmatrix}}_{\text{in } W^\perp}}$$

Check to make sure $\vec{z}_2 \in W^\perp$

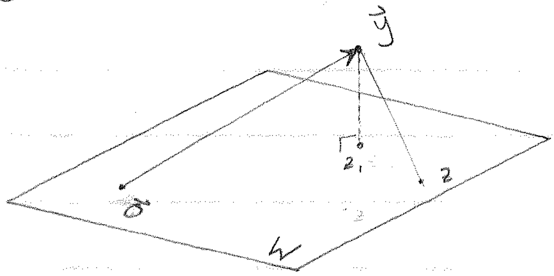
$$\left(c_1\begin{bmatrix}2\\0\\-3\end{bmatrix} + c_2\begin{bmatrix}6\\3\\4\end{bmatrix}\right)\cdot\begin{bmatrix}9\\-26\\6\end{bmatrix} = c_1\begin{bmatrix}2 & 0 & -3\end{bmatrix}\begin{bmatrix}9\\-26\\6\end{bmatrix} + c_2\begin{bmatrix}6 & 3 & 4\end{bmatrix}\begin{bmatrix}9\\-26\\6\end{bmatrix}$$

$$= c_1(0) + c_2(0) = 0$$ independent of c_1 & c_2.

Note that the Thm is always true, even if $\vec{y} \in W$ since then $\vec{z}_2 = \vec{0}$ which is in $W^\perp$ (orthogonal to every vector).

If $\vec{y} \in W^\perp$ then $\vec{z}_1 = \vec{0}$ but $\vec{0} \in W$ (zero vector is in every subspace of $\mathbb{R}^n$.)

6.3



Suppose we are constrained to remain in W. What element in W is the closest to $\vec{y}$?

The answer is... $\text{proj}_W \vec{y}$.

Let $\vec{z}_1 = \text{proj}_W \vec{y}$ and let $\vec{z}$ be any <u>other</u> vector in W. So that $\vec{z} - \vec{z}_1 \in W$ and $\vec{z} - \vec{z}_1 \neq \vec{0}$.

$$\vec{y} - \vec{z} = \vec{y} - \vec{z}_1 + \vec{z}_1 - \vec{z} = (\vec{y} - \vec{z}_1) + (\vec{z}_1 - \vec{z})$$

Since $\vec{y} - \vec{z}$ forms the hypotenuse of a right triangle (remember that $(\vec{y} - \vec{z}_1)$ and $(\vec{z}_1 - \vec{z})$ are orthogonal)

So

$$\|\vec{y} - \vec{z}\|^2 = \|\vec{y} - \vec{z}_1\|^2 + \|\vec{z}_1 - \vec{z}\|^2$$

Thus $\|\vec{y} - \vec{z}_1\| \leq \|\vec{y} - \vec{z}\|$ and $\|\vec{y} - \vec{z}_1\| = \|\vec{y} - \vec{z}\|$ iff $\vec{z}_1 = \vec{z}$

Thm If W is a subspace of $\mathbb{R}^n$, $\vec{z}$ is any element in W and $\vec{z}_1 = \text{proj}_W \vec{y}$. Then

$$\|\vec{y} - \vec{z}_1\| \leq \|\vec{y} - \vec{z}\|$$

and

$$\|\vec{y} - \vec{z}_1\| = \|\vec{y} - \vec{z}\| \quad \text{iff} \quad \vec{z} = \vec{z}_1$$

<u>Ex</u> Suppose $\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$, $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

Find the closest point to $\vec{y}$ in the subspace W.

This amounts to finding $\text{Proj}_W \vec{y}$

$$\text{Proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}}$$

Notice that if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for W then

$$\text{Proj}_W \vec{y} = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{y}\cdot\vec{u}_p)\vec{u}_p$$

$$= (\vec{u}_1^T\vec{y})\vec{u}_1 + \cdots + (\vec{u}_p^T\vec{y})\vec{u}_p$$

$$= [\vec{u}_1 \cdots \vec{u}_p]\begin{bmatrix} \vec{u}_1^T\vec{y} \\ \vdots \\ \vec{u}_p^T\vec{y} \end{bmatrix}$$

$$= UU^T\vec{y} \quad \text{if} \quad U = [\vec{u}_1 \cdots \vec{u}_p]$$