6.4 The Gram-Schmidt Process.

We have recently seen that there are distinct advantages to having an orthogonal basis. When we don't, how can we find one?

Given any basis of a subspace W of $\mathbb{R}^n$, we can compute an orthogonal basis.

"trade in"

Ex $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad W = \text{span}\{\vec{x}_1, \vec{x}_2\}$

1. Take $\vec{v}_1 = \vec{x}_1$ to be the first vector in our orthogonal basis

2. For the second vector, we want a vector in W but orthogonal to $\vec{v}_1$

$$\vec{v}_2 = \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1}_{\text{component of } \vec{x}_2 \text{ in direction of } \vec{v}_1}$$

is a vector orthogonal to $\vec{v}_1$

So
$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Thus, our orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\}$.

In fact, we can use any scalar multiple of these vectors in the basis so we can choose $\vec{v}_2' = 3\vec{v}_2$ as a basis vector to find $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

The process begun here is the Gram-schmidt orthogonalization process. Given any basis for a subspace of $\mathbb{R}^n$, we can (in theory) generate an orthogonal basis for this subspace.

Gram-Schmidt Process

0. Start with a basis $\{\vec{x}_1, \dots, \vec{x}_p\}$ of W, a subspace of $\mathbb{R}^n$.
1. $\vec{v}_1 = \vec{x}_1$
2. $\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \vec{v}_{p-1}$$

3. $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal basis for W.

Options: If we also normalize each $\vec{v}_i$ then $\vec{v}_i \cdot \vec{v}_i = 1$ and the algorithm becomes

1. $\vec{v}_1' = \vec{x}_1$; $\vec{v}_1 = \frac{\vec{v}_1'}{\|\vec{v}_1'\|}$

2. $\vec{v}_2' = \vec{x}_2 - (\vec{x}_2 \cdot \vec{v}_1)\vec{v}_1$; $\vec{v}_2 = \frac{\vec{v}_2'}{\|\vec{v}_2'\|}$

$$\vdots$$

$$\vec{v}_p' = \vec{x}_p - (\vec{x}_p \cdot \vec{v}_1)\vec{v}_1 - \cdots - (\vec{x}_p \cdot \vec{v}_{p-1})\vec{v}_{p-1} \quad ; \quad \vec{v}_p = \frac{\vec{v}_p}{\|\vec{v}_p\|}$$

3. $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthonormal basis.

Ex Find orthogonal basis for Col A if $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

$$\vec{V}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{V}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{\begin{bmatrix} -5 & 1 & 5 & -7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} 3 & 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{(-40)}{20}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{V}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & -2 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & -2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 & 3 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{(-10)}{20}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6/2 \\ 1 \\ 1 \\ 6/2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ forms an orthogonal basis for Col A.

Thm If A is an $m \times n$ matrix with lin. indep. columns then $A = QR$ where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col$A$ and R is an $n \times n$ upper triangular matrix with positive diagonal elements - thus R is invertible.

The matrix Q can be found column-by-column using the Gram-Schmidt (or some other) method.

How about R? Since Q has orthonormal columns, we know that $Q^TQ = I$, so

$$A = QR$$
$$Q^TA = Q^TQR = IR = R$$

<u>Ex</u> Factor $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$ into QR

To find columns of Q, just normalize the basis vectors found in the last example.

$$\vec{q}_1 = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{20}}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{q}_3 = \frac{1}{\sqrt{20}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{Q = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}}$$

$$R = Q^TA = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 & 1 & -1 & 3 \\ 1 & 3 & 3 & -1 \\ -3 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = \boxed{\frac{1}{\sqrt{20}}\begin{bmatrix} 20 & -40 & 30 \\ 0 & 20 & -10 \\ 0 & 0 & 20 \end{bmatrix} = R}$$