6.7 Inner Product Spaces

<u>Def</u> An <u>inner product</u> on a vector space V is a function that, to each pair of vectors $\vec{u}, \vec{v} \in V$, associates a real number $\langle \vec{u}, \vec{v} \rangle$ and satisfies the following axioms for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars c:

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$, $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$.

<u>Ex</u> Consider $\mathbb{P}_1$. Show that if $p, q, r \in \mathbb{P}_1$ then

$$\langle p, q \rangle = p(0)\,q(0) + p(1)\,q(1)$$

defines an inner product.

1. $\langle p, q \rangle = p(0)q(0) + p(1)q(1) = q(0)p(0) + q(1)p(1) = \langle q, p \rangle$
2. $\langle p+q, r \rangle = (p(0)+q(0))\,r(0) + (p(1)+q(1))\,r(1)$

$$= p(0)r(0) + q(0)r(0) + p(1)r(1) + q(1)r(1)$$
$$= p(0)r(0) + p(1)r(1) + q(0)r(0) + q(1)r(1)$$
$$= \langle p, r \rangle + \langle q, r \rangle$$

3. $\langle cp, q \rangle = cp(0)q(0) + cp(1)q(1)$

$$= c[p(0)q(0) + p(1)q(1)]$$
$$= c\langle p, q \rangle$$

4. $\langle p, p \rangle = p(0)p(0) + p(1)p(1) = p(0)^2 + p(1)^2 \geq 0$

We see that $p(0)^2 + p(1)^2 = 0$ iff $p(0) = p(1) = 0$

Which says that then p is the zero vector in $\mathbb{P}_1$

We define $\|\vec{v}\| = \sqrt{\langle \vec{v},\vec{v}\rangle}$ to be the "length" of the vector so that if

$$\|\vec{v}-\vec{u}_1\| < \|\vec{v}-\vec{u}_2\|$$

We say that (in some sense) $\vec{v}$ is closer to $\vec{u}_1$ than to $\vec{u}_2$

If $\langle \vec{u},\vec{v}\rangle = 0$ then $\vec{u}$ and $\vec{v}$ are orthogonal

Ex Show that $p=1$, $q=t$ are not orthogonal wrt the inner product in the last example.

$$\langle p,q\rangle = p(0)q(0) + p(1)q(1)$$
$$= 1\cdot 0 + 1\cdot 1 = 1 \neq 0$$

Now, consider the inner product

$$\langle p,q\rangle = p(-1)\,q(-1) + p(1)\,q(1)$$

Then

$$\langle p,q\rangle = (1)(-1) + (1)(1) = -1+1 = 0$$

So p and q are orthogonal wrt this new inner product.

Be sure of your Inner Product!

Ex. Suppose $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$ is an inner product defined for $\mathbb{P}_2$.

Find the orthogonal projection of $p(t) = 1 + t - t^2$ onto $W = \text{span}\{1, t\}$

$$\text{Proj}_W p = \frac{\langle p, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle p, t\rangle}{\langle t, t\rangle} t$$

$$\langle p, 1\rangle = p(-1)\cdot 1 + p(0)\cdot 1 + p(+1)\cdot 1$$
$$= (-1) + 1 + 1 = 1$$
$$\langle 1, 1\rangle = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3$$

$$\langle p, t\rangle = (-1)(-1) + (1)(0) + (1)(1) = 2$$
$$\langle t, t\rangle = (-1)(-1) + (0)(0) + (1)(1) = 2$$

$$\text{Proj}_W p = \frac{1}{3}\cdot 1 + \frac{2}{2} t = \frac{1}{3} + t$$

So $q = p - \left(\frac{1}{3} + t\right) \quad 1 + t - t^2 - \frac{1}{3} - t = \frac{2}{3} - t^2$

Should be orthogonal to both 1 and t.

$$\langle q, 1\rangle = \left(\tfrac{2}{3} - 1\right)(1) + \left(\tfrac{2}{3} - 0\right)(1) + \left(\tfrac{2}{3} - 1\right)(1)$$
$$= -\frac{1}{3} + \frac{2}{3} - \frac{1}{3} = 0 \checkmark$$

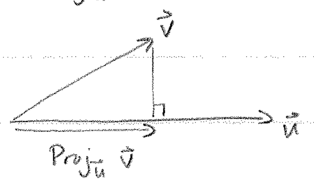
$$\langle q, t\rangle = \left(\tfrac{2}{3} - 1\right)(-1) + \left(\tfrac{2}{3} - 0\right)(0) + \left(\tfrac{2}{3} - 1\right)(+1)$$
$$= +\frac{1}{3} + 0 - \frac{1}{3} = 0 \checkmark$$

Thus $\hat{p} = \mathrm{Proj}_W p = \frac{1}{3} + t$ is the "best" approximation (i.e. the closest vector) to p with respect to this inner product.

If $\mathrm{Proj}_{\vec{u}} \vec{v}$ means the orthogonal projection of $\vec{v}$ onto the space spanned by $\vec{u}$, then

$$\| \mathrm{Proj}_{\vec{u}} \vec{v} \| = \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} \right\| = \left| \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{u}, \vec{u} \rangle} \right| \|\vec{u}\| = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\|^2} \|\vec{u}\| = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\|}$$

Q: is $\| \mathrm{Proj}_{\vec{u}} \vec{v} \| \le \| \vec{v} \|$? Yes, by Pythagorean Thm



$$\| \mathrm{Proj}_{\vec{u}} \vec{v} \|^2 + \| \vec{v} - \mathrm{Proj}_{\vec{u}} \vec{v} \|^2 = \| \vec{v} \|^2$$

So we have $\frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\|} = \| \mathrm{Proj}_{\vec{u}} \vec{v} \| \le \| \vec{v} \|$

$$\boxed{\therefore \quad |\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\| \, \|\vec{v}\|}$$

Cauchy-Schwartz Inequality

Now

$$\begin{aligned}
\| \vec{u} + \vec{v} \|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle \\
&= \langle \vec{u} + \vec{v}, \vec{u} \rangle + \langle \vec{u} + \vec{v}, \vec{v} \rangle \\
&= \langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\
&= \|\vec{u}\|^2 + 2 \langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\
&\le \|\vec{u}\|^2 + 2 |\langle \vec{u}, \vec{v} \rangle| + \|\vec{v}\|^2 \\
&\le \|\vec{u}\|^2 + 2 \|\vec{u}\| \, \|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2 \quad \rightarrow
\end{aligned}$$

Triangle Inequality

$$\boxed{\| \vec{u} + \vec{v} \| \le \|\vec{u}\| + \|\vec{v}\|}$$