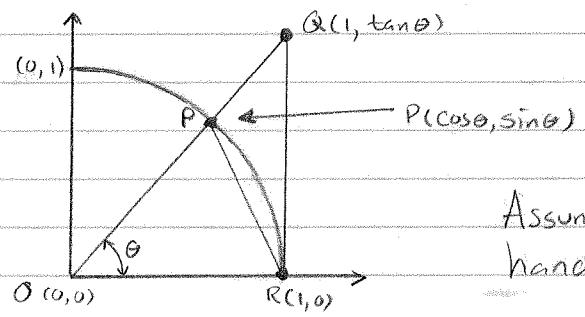
Derivatives of Trigonometric Functions

Before we start to find derivatives of the trigonometric functions, we need to investigate a few limits.

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ We've been using 1 for this limit but we have not yet proved that this is the correct value.

$$\boxed{\text{Lemma (a supporting theorem)} \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Proof:



This proof uses a clever geometric construction and the squeeze theorem.

Assume $0 < \theta < \pi/2$. We'll return and handle $-\pi/2 < \theta < 0$ shortly.

Notice that $0 < \Delta OPR < \text{sector } OPR < \Delta OQR$

$$\text{Sector } OPR = (\text{area of circle}) \cdot \frac{\theta}{2\pi} = \pi(1)^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$$

$$\Delta OPR = \frac{1}{2}(1)(\sin\theta) = \frac{\sin\theta}{2}$$

$$\Delta OQR = \frac{1}{2}(1)(\tan\theta) = \frac{\tan\theta}{2}$$

$$\therefore \quad 0 < \frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$

Since $\frac{\sin\theta}{2} > 0$, we can divide by it to obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

Each of these quantities is positive so

$$1 > \frac{\sin\theta}{\theta} > \frac{\sin}{\tan\theta}$$

but $\sin\theta/\tan\theta = \sin\theta/(\sin\theta/\cos\theta) = \cos\theta$

So we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Taking limits, we have $\lim_{\theta\to 0} 1 = 1$, $\lim_{\theta\to 0} \cos\theta = 1$

where the second of these is true because $\cos\theta$ is a continuous function.

By the squeeze theorem we have $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$.

Now, back to $-\pi/2 < \theta < 0$. In this case we have

$$0 > \frac{\sin\theta}{2} > \frac{\theta}{2} > \frac{\tan\theta}{2}$$

since all the corresponding areas are below the y axis.

As before, we divide by $\sin\theta/2$, but since this is negative we need to change the direction of the inequalities

$$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

or

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Picking up with the limits above and using the squeeze theorem once more we find that $\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$.

$\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ ✓

$$\boxed{\text{Lemma} \quad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.}$$

Proof $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$

$$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta}$$

$$= (1) \cdot \frac{0}{1+1}$$

$$= 0.$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}$$

$$= (\sin x)\lim_{h\to 0}\frac{\cos h-1}{h}+(\cos x)\lim_{h\to 0}\frac{\sin h}{h}$$

$$= (\sin x)\cdot 0+(\cos x)\cdot 1 = \cos x$$

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

A similar derivation shows

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\cos x)(\sin x)'-(\sin x)(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x-(-\sin^2 x)}{\cos^2 x}$$

$$= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

$$\boxed{\frac{d}{dx}\cot x = -\csc^2 x}$$

$$\boxed{\frac{d}{dx}\sec x = \sec x\tan x}$$

$$\boxed{\frac{d}{dx}\csc x = -\csc x\cot x}$$

Derivatives of Trigonometric Functions

Ex $\frac{d}{dx} \sin^2 x = \frac{d}{dx} (\sin x)^2$

$= 2(\sin x)(\cos x) = 2 \sin x \cos x$

Ex $\frac{d}{dx} \cos(x^2) = -\sin(x^2) \cdot (2x) = -2x \sin x^2$

Ex $\frac{d}{dx} \tan\sqrt{1+x} = \sec^2\sqrt{1+x} \cdot \frac{1}{2}(1+x)^{-1/2} \cdot 1 = \frac{\sec^2\sqrt{1+x}}{2\sqrt{1+x}}$

Ex $\frac{d}{dx} x^3 \sec x = 3x^2 \sec x + x^3 \sec x \tan x = x^2 \sec x\,(3 + x \tan x)$

Ex $\frac{d}{dx} \frac{\cos x}{x^2} = \frac{x^2(-\sin x) - 2x\cos x}{x^4} = -\frac{x \sin x + 2\cos x}{x^3}$

Ex $\frac{d}{dx} \sin\left(\frac{x^2}{1-x}\right) = \cos\left(\frac{x^2}{1-x}\right)\left[\frac{(1-x)2x - (-1)x^2}{(1-x)^2}\right]$

$= \cos\left(\frac{x^2}{1-x}\right)\left(\frac{2x - 2x^2 + x^2}{(1-x)^2}\right)$

$= \cos\left(\frac{x^2}{1-x}\right) \frac{2x - x^2}{(1-x)^2}$

Ex. Find the equation of the line tangent to $\cos x$ at $x = \pi/2$

$\frac{d}{dx} \cos x = -\sin x \rightarrow \frac{d}{dx} \cos x \Big|_{x=\pi/2} = -\sin(\pi/2) = -1$

Slope $= -1$, point $= (\pi/2, 0)$

$y - 0 = (-1)(x - \pi/2) = \boxed{y = -x + \pi/2}$